COMPSCI 240: Reasoning Under Uncertainty

Andrew Lan and Nic Herndon

University of Massachusetts at Amherst

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Lecture 8: Discrete random variables, and Probability mass functions

Random Variable

A new concept
But you already learned it.

Review Experiments, Sample Spaces, Events

- *Experiment:* a process that results in exactly one of several possible outcomes, e.g., rolling a dice
- *Sample space:* the set of all possible outcomes of an experiment, e.g., $\Omega = \{1, 2, 3, 4, 5, 6\}$
- *Event:* a subset of Ω, e.g., $A =$ "odd number" $= \{1, 3, 5\}$
- *Atomic event:* event consisting of a single outcome, e.g., $\{1\}$

$$P(A) = P(\{o_1\}) + ... + P(\{o_N\}) \ .$$

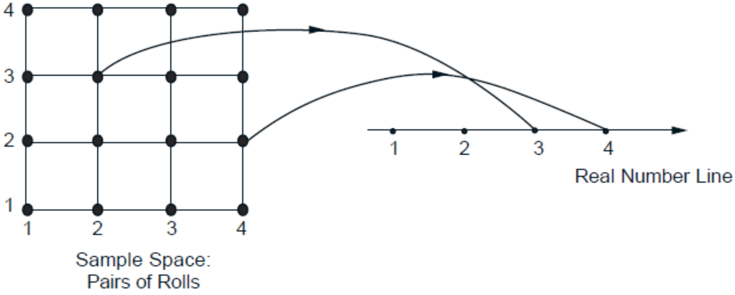
Random Variable

- A *random variable* is a real-valued function that maps from the sample space to the real numbers,

$$X : \Omega \to \mathbb{R}$$

Example: Maximum of Dice Rolls

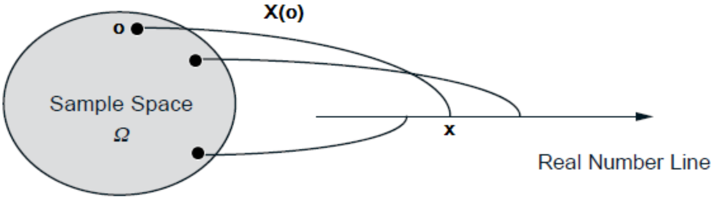
- Consider rolling two fair four sided dice.
- The outcomes $r \in \Omega$ are pairs $r = (r_1, r_2)$ where r_1, r_2 both take values on the set $\{1, ..., 4\}$
- We could consider a function $X(r_1, r_2) = \max(r_1, r_2)$ a random variable.



Other Examples of Random Variable

- The sum of the two rolls.
- The number of ones in the two rolls.
- The second roll raised to the fifth power.
- Height of a randomly selected student in CS240.
- Temperature of the CS240 lecture hall.

More Generally...



- For every outcome $o \in \Omega$, a random variable defines a single real number

$$X(o) \in \mathbb{R} \ .$$

- Note that it is possible to have multiple outcomes $o_1, o_2, \ldots$ such that $X(o_1) = X(o_2) = \ldots$.

Random Variables Give An Easy Way to Specify Events

- If we have a function $X : \Omega \to \mathbb{R}$, we can use it to construct a different event for each value of $x \in \mathbb{R}$:

$$\{X = x\} = \{o | o \in \Omega \text{ and } X(o) = x\}$$

- In the dice example, the event $\{X = x\}$ is the set of outcomes $o \in \Omega$ that are mapped to the the same value x by the function X.
  For example,

$$\{X = 2\} = \{(1, 2), (2, 1), (2, 2)\}$$

$$\{X = 3\} = \{(1, 3), (2, 3), (3, 3)(3, 1), (3, 2)\}$$

$$\{X = 1\} = \{(1, 1)\}$$

Discrete Random Variables and Probability

- A random variable is called **discrete** if its input (sample space) is either finite or countably infinite.
- We can compute the probability of an event $\{X = x\}$ by decomposing it into atomic events and using the probability rule:

$$P(X = x) = p_X(x) = P(\{o | o \in \Omega \text{ and } X(o) = x\})$$

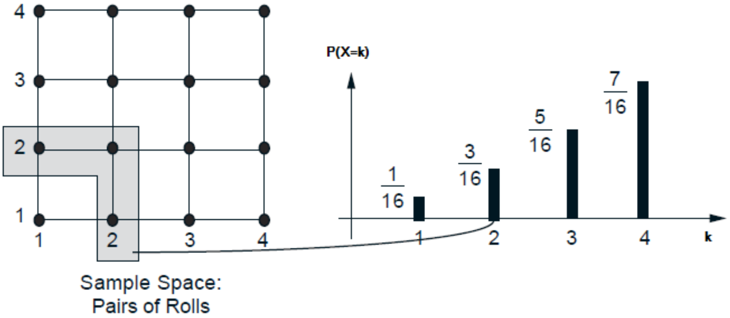
- *Probability law:* A function $p_X(x)$ that maps event to a number between 0 and 1 that satisfies the probability axioms:
  1. *Nonnegativity:* $p_X(x) \geq 0, \forall x$.
  2. *Normalization:* $\sum_x p_X(x) = 1$.

Example: Maximum of Dice Rolls

- For example, in the event of $\{X = 2\}$ for the dice rolling example where $X(r_1, r_2) = \max(r_1, r_2)$

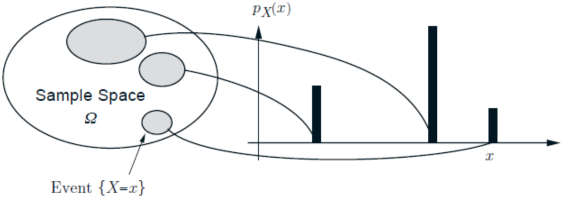
$$\begin{aligned} P(X = 2) &= P(\{(1,2),(2,1),(2,2)\}) \\ &= P((1,2)) + P((2,1)) + P((2,2)) = 3/16 \end{aligned}$$

- We can work out the probability for all possible values of x:



In general. . .

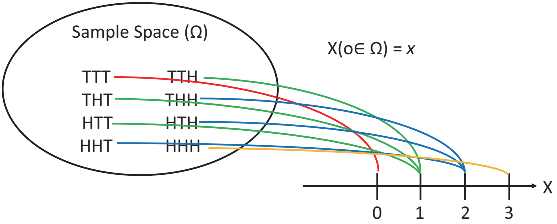
- The probability associated with the event $\{X = x\}$ for each element $x \in \mathbb{R}$ of a discrete random variable X is referred to as the **probability mass function** or **PMF** of the random variable.
    - Why do you call it a mass?
- The PMF is denoted by $P(X = x)$ or $p_X(x)$.



- The x-axis represents all possible outcomes of the event
- The y-axis represents the associated probabilities

Discrete Random Variable Example

- Consider tossing a coin $n = 3$ times.
- At each toss, the coin comes up a head with probability $p = \frac{1}{2}$, and a tail with probability $1 - p = \frac{1}{2}$.
- **Q**: Is the number of heads in the sequence a random variable?
- **A**: Yes. It converts experiment outcomes (the sample space) into real-valued numbers.



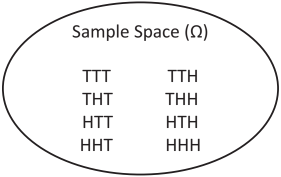
Discrete Random Variable Example

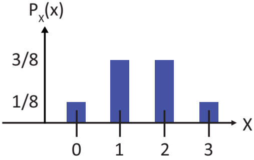
- Can we mathematically define the PMF?

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P_X(x) = \binom{3}{x} \frac{1}{2}^x \frac{1}{2}^{3-x}$$

- How does the PMF look as a function of x.





More Discrete Random Variable Example...

Suppose you have a fair coin that shows a value 1 on one side and value -1 on the other side. Let X represent the number that you see after a single flip. What does PMF look like?

More Discrete Random Variable Example...

The discrete random variable X can take only the value 1, 2, and 3. Suppose that the PMF is defined by

$$p_X(x) = \frac{x^2 + k}{50}$$

Then, what is the value of k?