COMPSCI 240: Reasoning Under Uncertainty

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Lecture 7: Counting (cont.)

Summary of Counting Problems

Structure	Description	Order Matters	Formula
Permutation	Number of ways to order n objects	Yes	$n!$
k-Permutation	Number of ways to form a sequence of size k using k different objects from a set of n objects	Yes	$\frac{n!}{(n-k)!}$
Combination	Number of ways to form a set of size k using k different objects from a set of n objects	No	$\frac{n!}{k!(n-k)!}$
Partition	Number of ways to partition n objects into ℓ groups of size $n_1, \ldots, n_\ell$	No	$\frac{n!}{n_1! \ldots n_\ell!}$

Example: Grade Assignments

- **Question:** Suppose a professor decides at the beginning of the semester that in a class of 10 students, 3 A's, 4 B's, 2 C's and one C- will be given. How many different ways can the students be assigned grades at the end of the semester?
- **Answer:** This is a partition problem. There are 10 objects and 4 groups. The group sizes are 3,4,2,1. The answer is thus:

$$\frac{10!}{3! \cdot 4! \cdot 2! \cdot 1!}$$

Example: Top of the Class

- **Question:** A computer science program is considering offering three senior year scholarships to their top three incoming seniors worth $\$10,000$, $\$5,000$ and $\$2,000$. If there are 100 incoming seniors, how many ways are there for the scholarships to be awarded?
- **Answer:** This is a k-permutation problem. There are 100 students and 3 distinct scholarships. The number of assignments of students to scholarships is thus:

$$\frac{100!}{(100-3)!} = 100 \cdot 99 \cdot 98$$

Example: Exam Seating

- **Question:** For the final exam in a class of 16 students, the professor arranges the desks in 4 rows of 4 desks each. How many ways can the students be assigned to desks for the exam?
- **Answer:** This is a permutation problem. There are 16 students to assign to 16 desks. The spatial arrangement of the desks is irrelevant. The answer is thus:

$$16!$$

Example: Overbooked

- **Question:** Suppose a class with 50 students is enrolled in a room with only 40 seats. Unfortunately, the poor professor has to take only 40 students in the class. How many ways are there for 40 of the 50 students to get a seat in the class?
- **Answer:** This is a combination problem. We only care if a student gets a seat, not which seat they get. The answer is thus:

$$\binom{50}{40} = \frac{50!}{40!10!}$$

Example: Independent Coin Flips

- Suppose we have a biased coin that lands heads with probability p and tails with probability $(1-p)$, where $0 < p < 1$.
- If we flip this coin 3 times, what is the probability that the outcome is the **sequence** HTH?
- Let H_i be the event that the ith coin flip is heads. By independence,

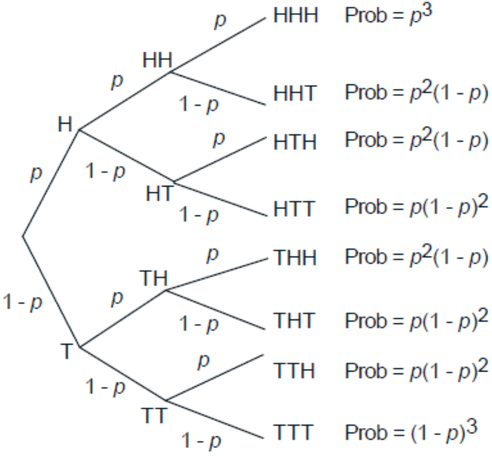
$$P(HTH) = P(H_1 \cap H_2^c \cap H_3) = P(H_1)P(H_2^c)P(H_3)$$

  which is simply:

$$p \cdot (1-p) \cdot p = p^2(1-p)$$

Example: Independent Coin Flips

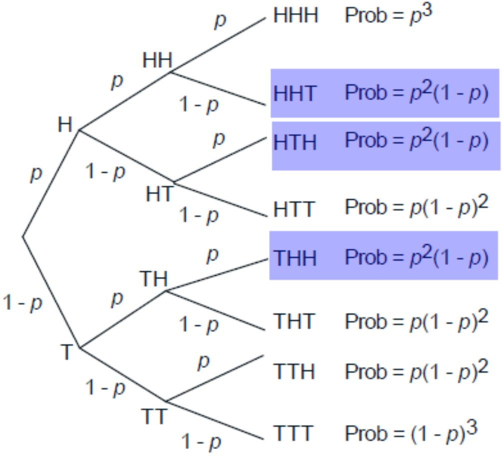
- If we flip the coin 3 times, what is the probability that the **number** of heads in the outcome is 2?
- It's helpful to use a conditional probability tree.



Example: Independent Coin Flips

- Probability of two heads?

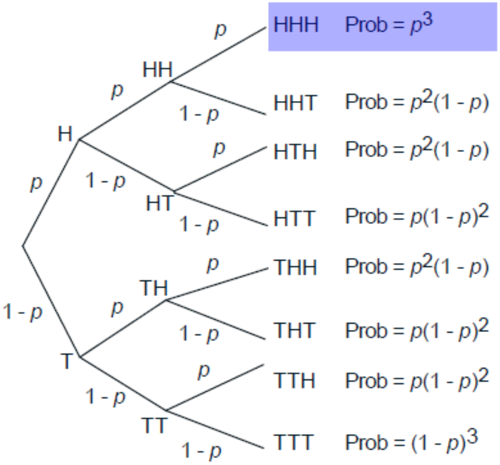
$$3 \cdot p^2(1 - p)$$



Example: Independent Coin Flips

- Probability of three heads?

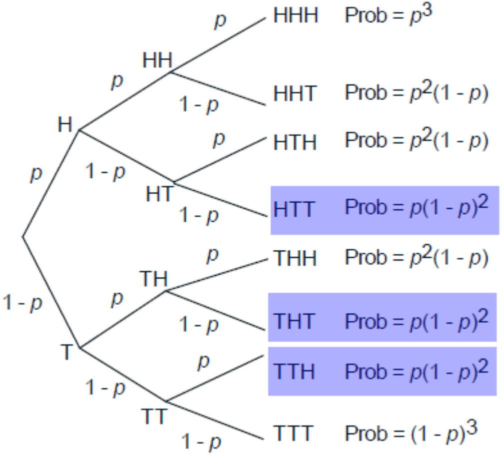
$$p^3$$



Example: Independent Coin Flips

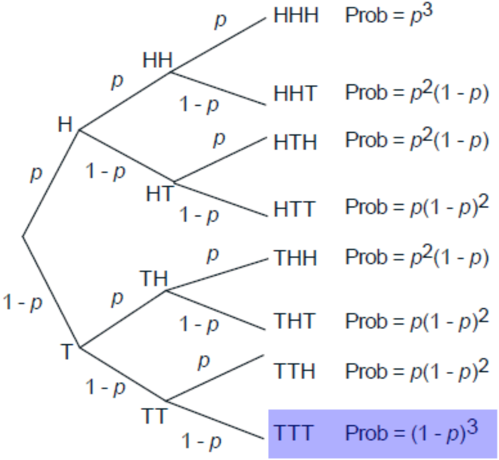
- Probability of one head?

$$3 \cdot p(1-p)^2$$



Example: Independent Coin Flips

- Probability of no heads? $(1-p)^3$



Example: Independent Coin Flips

- Since we only flip the coin three times, the four events $3H$, $2H$, $1H$, $0H$ partition the outcome space. By finite additivity, their probabilities should add to one.

$$P(3H) = p^3 \qquad P(2H) = 3p^2(1-p)$$
$$P(1H) = 3p(1-p)^2 \qquad P(0H) = (1-p)^3$$

$$\begin{aligned}
P(\Omega) &= P(3H) + P(2H) + P(1H) + P(0H) \\
&= (p^3) + (3p^2(1-p)) + (3p(1-p)^2) + ((1-p)^3) \\
&= (p^3) + (3p^2 - 3p^3) + (3p - 6p^2 + 3p^3) + (1 - 3p + 3p^2 - p^3) \\
&= 1
\end{aligned}$$

Generalizing to n Flips and k Heads: The Binomial Law

- If we toss n coins, what's the probability of seeing k heads, denoted as $P_n(k)$? (without exhaustively enumerating all sequences?)
- Any single sequence of length n with k heads has probability
$$p^k(1-p)^{n-k} \; .$$
- But how many different sequences of length n contain k heads?
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
where $\binom{n}{0} = 1$.
- Thus,
$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Generalizing to n Flips and k Heads: The Binomial Law

- The following equation is often called the **binomial probabilities**.

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where $\binom{n}{k}$ is referred to as **binomial coefficient**.

Application: Redundancy and Networks

Question: Suppose you're responsible for the network of web servers that host a large website. If each server has an independent chance of failure of 0.1% per day and you have 1000 servers, what is the probability that no servers will fail on any given day?
Answer: We have $p = 0.001$, $n = 1000$ and $k = 0$. The required probability is thus:

$$P_{1000}(0) = \binom{1000}{0} 0.001^0 (0.999)^{1000} = 0.3677$$