COMPSCI 240: Reasoning Under Uncertainty

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Lecture 6: Counting

Revisit: Discrete Probability Laws

- If Ω is finite and all outcomes are equally likely, then

$$P(A) = \frac{|A|}{|\Omega|}$$

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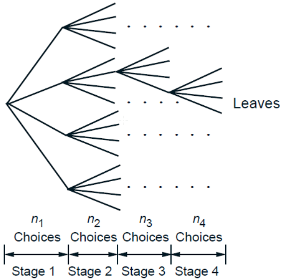
- The calculation of probabilities often involve **counting** the number of outcomes in various events.
- Sometimes it's challenging to compute $|A|$ and $|\Omega|$ and they are too large work out by hand...

We are going to learn about different counting methods:

- *Permutations*
- *k-Permutations*
- *Combinations*
- *Partitions*

The Counting Principle

- Consider a sequential process with s stages. At each stage i, there are n_i possible results. How many outcomes does the process have?



- How many possible outcomes are possible from a sequence of s stages?

$$n_1 \times n_2 \times \cdots \times n_s = \prod_{i=1}^{s} n_i.$$

Example: Phone Numbers

- **Question:** How many different 7 digit phone numbers are there?
- **Answer:** This is an $s = 7$ stage experiment with $n_i = 10$ possible events per stage. This gives 10^7 possible phone numbers.
- **Question:** If your new cell number is randomly assigned, what's the probability that the last two digits match the day of your birth date?
- **Answer:** If the last two digits are your birth "day" (e.g. 05, 31, etc.), then there is only one choice for these two digits and 10^5 choices for the remaining 5 digits. The probability is thus $10^5/10^7$ or 1 in 100.

Counting Permutations

- Let S be a set of n objects.
- Consider an n-stage experiment where at each stage we choose one object without replacement.
  - We pick objects until there's no more objects to pick.
- This process produces an **ordering** or **permutation** of the n objects.
  - For example, if $n = 3$ and $S = \{a, b, c\}$, one ordering can be bac.
- This is an n stage process. We have $s_1 = n$, $s_2 = n - 1$,..., $s_n = 1$.
- By the counting principle, the number of permutations is

$$n(n-1)(n-2)\cdots 1 = n!$$

- For permutations, order matters, i.e., $abc \neq bac$.

Counting k-Permutations

- Let S be a set of n objects.
- Consider a k-stage experiment where $k \leq n$. At each stage we choose one object without replacement.
  - We pick only k objects.
- This process produces an ordering of the k objects, which is also called a k-permutation.
  - For example, if $n = 3, k = 2$, and $S = \{a, b, c\}$, one possible 2-permutation is ba and another is ab.
- This is a k-stage process where $s_1 = n$, $s_2 = n - 1$,…, $s_k = n - k + 1$.
- By the counting principle, the number of permutations is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- Order also matters for k-permutations.

Example: A 100-meter Race

- There are 8 contestants racing for a 100-meter race. The contest provides three different medals: Gold, Silver, Bronze.
- How many different medal orderings are there?

- This is a $k = 3$ stage process with $n = 8$ objects. Since the ordering matters, this is a k-permutation problem. The answer is thus $8!/(8-3)! = 336$.

Example: A 100-meter Race

- If all medal orderings are equally likely, what's the probability that Bolt (one of the contestants) gets a medal?
- If Bolt takes the first spot, there are $7!/(7-2)!$ ways the other 7 runners could be assigned to the remaining spots. This is also true if Bolts takes the second or third spots.
- So, final answer is:

$$\frac{3 \times \frac{7!}{5!}}{\frac{8!}{5!}} = \frac{3}{8}$$

Counting Combinations

- Let S be a set of n objects. How many subsets of size k are there?
- The number of k-permutations is $n!/(n-k)!$ but this over counts the number of subsets, e.g., ab and ba are different 2-permutations of $\{a, b, c\}$, but the same subset $\{a, b\}$.
  - Order does **NOT** matters for combinations.
- $k!$ different k-permutations belong to the same subset of k objects, so the number of "k-combinations" is

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!},$$

  which is denoted $\binom{n}{k}$, pronounced as "n choose k".
- Note that $\binom{n}{0} = 1$

Example: A 100-meter Race

- There are 8 contestants racing for a 100-meter race. The contest provides three different medals: Gold, Silver, Bronze.
- How many different medalist groups are there?
  - Note that our previous question was "How many different medal orderings are there?"

- This is a $k = 3$ stage process with $n = 8$ objects. Since the ordering does not matter, this is a combination problem.
- The answer is thus

$$\binom{8}{3} = \frac{8!}{(8-3)!3!} = 56.$$

Example: Netflix

- **Question:**Netflix streams 6000 movies. You want to watch 3 of them. How many different movie combinations can you choose to watch?
- **Answer:** Since the order in which you watch the movies does not matter, this is a combination problem. The number of possible choices is $\binom{6000}{3}$ or 35,982,002,000 .

Example: Netflix

- **Question:** Suppose you randomly choose 3 movies and you will be SAD if you end up watching *Ghost* starring Demi Moore and Patrick Swayze. What is your probability of being SAD? What is the probability that you are NOT SAD?
- **Answer:** You will be SAD if one of your choices is *Ghost*. The remaining two movies can be chosen in $\binom{5999}{2}$ or 17,991,001 ways. Hence the probability that you are SAD is

$$\frac{17991001}{35982002000} = 0.0005$$

  On the other hand, you will be NOT SAD if all 3 of your movies are chosen from the 5999 movies. There are $\binom{5999}{3}$ or 35,964,010,999 ways to do that. The probability you are HAPPY is

$$\frac{35964010999}{35982002000} = 0.9995$$

One Challenging Counting Example

- How many ways are there for 3 men and 3 women to stand in a line so that no two women stand next to each other?
- First, we figure out how many ways to arrange the 3 men in a line.
  - Number of permutation of 3 men is $3! = 6$.
- Second, for each arrangement of men, there are 4 slots to place the 3 women so that they do not stand next to each other, which is $\binom{4}{3}$.
  - W M W M W M
  - M W M W M W
  - W M W M M W
  - W M M W M W
- Third, for each combination of women, we can then permute the arrangement of women
  - Number of permutation of 3 women is $3! = 6$.
- Thus, total is

$$3! \times \binom{4}{3} \times 3! = 144$$

Counting Partitions

- A combination divides items into one group of k and one group of $n - k$. Thus, a combination can be viewed as a partition of the set in two.
- Consider an experiment where we divide n objects into ℓ groups with sizes $n_1, n_2, ..., n_\ell$ such that $n = \sum_{i=1}^{\ell} n_i$.
- How many partitions are there?
- There are $\binom{n}{n_1}$ ways to choose the objects for the first partition. This leaves $n - n_1$ objects. There are $\binom{n-n_1}{n_2}$ ways to choose objects for the second partition. There are $\binom{n-n_1-n_2-...-n_{\ell-1}}{n_\ell}$ ways to choose the objects for the last group.

Counting Partitions

- Using the counting principle, the number of partitions is thus:

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdots \binom{n-n_1-n_2-...-n_{\ell-1}}{n_\ell}$$

$$= \frac{n!}{n_1!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n-n_1-n_2-...-n_{\ell-1})!}{n_\ell!(n-n_1-n_2-...-n_\ell)!}$$

- Note that $(n-n_1-n_2-...-n_\ell)! = 0! = 1$.
- Canceling terms yields the final result:

$$\frac{n!}{n_1! \cdots n_\ell!}$$

Example: Discussion Groups

- **Question:** How many ways are there to split a discussion section of 12 students into 3 groups of 4 students each?
- **Answer:** This is a partition problem with 3 partitions of 4 objects each from a total of 12 objects. Using the partition counting formula, the answer is:

$$\frac{12!}{4! \cdot 4! \cdot 4!} = \frac{12!}{(4!)^3} = 34,650$$

Summary of Counting Problems

Structure	Description	Order Matters	Formula
Permutation	Number of ways to order n objects	Yes	$n!$
k-Permutation	Number of ways to form a sequence of size k using k different objects from a set of n objects	Yes	$\frac{n!}{(n-k)!}$
Combination	Number of ways to form a set of size k using k different objects from a set of n objects	No	$\frac{n!}{k!(n-k)!}$
Partition	Number of ways to partition n objects into ℓ groups of size $n_1, \ldots, n_\ell$	No	$\frac{n!}{n_1! \ldots n_\ell!}$