COMPSCI 240: Reasoning Under Uncertainty

Nic Herndon and Andrew Lan

University of Massachusetts at Amherst

Spring 2019

Lecture 5: Independence

Homework #1

- Homework #1 will be available on Moodle by 4 PM today (February 1, 2019).
- You will have to submit it to Gradescope by 4 PM next Friday (Febrarury 8, 2019).
- Good news: homework #1 is relatively easier than the rest of the homeworks.

From Last Class: Bayes' Rule

Let $A_1, A_2, \ldots, A_n$ partition Ω and $P(A_i) > 0$. For any B such that $P(B) > 0$,

$$
\begin{aligned}
P(A_i|B) &= \frac{P(A_i)P(B|A_i)}{P(B)} \\
&= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}
\end{aligned}
$$

Bayes' rule is often used for **inference**.

- What are the **causes** of the **observations** that we are making?

The events $A_1, A_2, \ldots, A_n$ can represent the causes (e.g., having disease or not) and B represents the observations (e.g., lab test results).
Given observations B, we are primarily interested in finding out the most likely cause A_i (i.e., A_i that maximizes $P(A_i|B)$).

A New Concept: Independence

- Consider flipping a fair coin twice in a row.
- If we know the coin is fair, does knowing the result of the first flip give us any information about the result of the second flip?
- What's the probability the coin comes up heads on the second flip?
- What's the probability the coin comes up heads on the second flip given that it came up heads on the first flip?

Probabilistic Independence

- Intuitively, when knowing that one event occurred doesn't change the probability that another event occurred or will occur, we say that the two events are *probabilistically independent*.
- We say that two events A and B are independent if and only if (iff)

$$P(A \cap B) = P(A)P(B) \ .$$

and this implies that $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

Rolling Two Dice

- **Question:** Suppose you roll two fair four sided dice. Is the event $A =$ "first roll is 3" independent of the event $B =$ "second roll is 4"?
- **Answer 1:** Intuitively, like the coin flip, the two rolls have nothing to do with each other so the events A and B should be independent.
- **Answer 2:** Formally, $P(A \cap B) = 1/16$ since there are 16 possible outcomes and the event $A \cap B$ refers to exactly one of them. $P(A) = 1/4$ since there's a 1/4 chance that the first roll is a 3. Similarly, $P(B) = 1/4$. Thus,

  $$P(A)P(B) = (1/4)(1/4) = 1/16$$

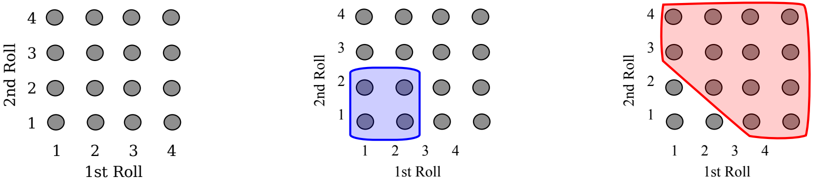
  so the events are independent.
- You are expected to answer homework/exam questions using the form of **Answer 2**.

Rolling Two Dice

- **Question:** Suppose you roll two fair four sided dice. Is the event $A =$"maximum is less than 3" independent of the event $B=$"sum is greater than 3"?
- **Answer 1:** Intuitively, the answer is no. If the maximum was low it would appear that this should reduce the probability of the sum being greater than 3.
  - $P(B|A) \neq P(B)$.

Rolling Two Dice

- **Question:** Suppose you roll two fair four sided dice. Is the event $A =$"maximum is less than 3" and $B=$"sum is greater than 3"?



Sample Space $\quad |A| = 4 \quad |B| = 13$

- **Answer 2:** Formally,

$$P(A \cap B) = \frac{1}{16}\ ,\ P(A) = \frac{1}{4} \text{ and } P(B) = \frac{13}{16}\ .$$

Since $\frac{1}{16} \neq \frac{1}{4} \cdot \frac{13}{16}$, the events are not independent.

An Event and Its Complement

- **Question:** Are A and A^c independent if $0 < P(A) < 1$?
- **Answer 1:** Intuitively, no. If you know A happens, then you know A^C does not happen.
- **Answer 2:** Formally, $P(A \cap A^C) = P(\emptyset) = 0$. If $0 < P(A) < 1$, then

$$P(A)P(A^C) \neq 0 \ .$$

Independence of Three Events

- Three events A, B, and C are independent iff:

$$P(A \cap B) = P(A)\,P(B)$$
$$P(A \cap C) = P(A)\,P(C)$$
$$P(B \cap C) = P(B)\,P(C)$$
$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

- First three conditions imply that any two events are independent (known as *pairwise independence*)
- *Pairwise independence* does not imply the *independence of all events*.
- Suppose we have a finite collection of events $A_1, A_2, ..., A_n$. These events are said to be independent iff

$$P\left(\cap_{i \in \mathcal{S}} A_i\right) = \prod_{i \in \mathcal{S}} P(A_i), \text{ for every subset } \mathcal{S} \text{ of } \{1, 2, ..., n\}$$

Chevalier de Méré Problem

Gamblers in the 1717 France rolled two die 24 times with a bet on having at least one double 6.

de Méré lost money.

A 17th century mathematician and poor gambler, Chevalier de Méré, made it to history by turning to Blaise Pascal for an explanation of his unexpected losses. Pascal combined his efforts with his friend Pierre de Fermat and the two of them laid out mathematical foundations for the theory of probability.

Chevalier de Méré Problem

- A_i = Throw i shows two 6
- $P(A_i) = \frac{1}{36}; \quad P(A_i^c) = \frac{35}{36}$
- Probability of de Méré losing (having A_i^c for all 24 trials):

$$P(\cap_{i=1}^{24} A_i^c) = \left(\frac{35}{36}\right)^{24} = 0.5086$$

- Probability of de Méré winning $= 1 - 0.5086 = 0.4914$.

Conditional Independence

- A and B are *conditionally independent* given C iff

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

- This is equivalent to $P(A \mid B \cap C) = P(A \mid C)$, assuming that $P(B \mid C) > 0$.
  - If C is given, additional information of knowing B has occurred does not change the conditional probability of A.
- This is equivalent to $P(B \mid A \cap C) = P(B \mid C)$, assuming that $P(A \mid C) > 0$.
  - If C is given, additional information of knowing A has occurred does not change the conditional probability of B.

Conditional Independence Example

- I have one fair coin and one biased coin that lands heads with probability $2/3$.
- I pick a coin with equal probability: let F be the event it's the fair coin and let F^c be the event it's the biased coin.
- I toss the chosen coin twice.
- Let A be the event the first toss is heads.
- Let B be the event the second toss is heads.
- Are A and B independent?

Conditional Independence Example

- Compute $P(A)$ and $P(B)$:

$$\begin{aligned} P(A) &= P(A|F)P(F) + P(A|F^c)P(F^c) \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} = \frac{7}{12} \\ &= P(B) \end{aligned}$$

  using the Total Probability Theorem.

- Compute $P(A \cap B)$:

$$\begin{aligned} P(A \cap B) &= P(A \cap B|F)P(F) + P(A \cap B|F^c)P(F^c) \\ &= \frac{1}{4} \times \frac{1}{2} + \frac{4}{9} \times \frac{1}{2} = \frac{25}{72} \end{aligned}$$

- $P(A \cap B) \neq P(A)P(B)$, so $P(A)$ and $P(B)$ are NOT independent.

Conditional Independence Example

- Are A and B conditionally independent given F?
- Compute $P(A|F)$ and $P(B|F)$:

$$P(A|F) = P(B|F) = 1/2$$

- Compute $P(A \cap B|F)$:

$$P(A \cap B|F) = 1/4$$

- $P(A \cap B|F) = P(A|F)P(B|F)$, so $P(A)$ and $P(B)$ are conditionally independent given F.