# COMPSCI 240: Reasoning Under Uncertainty

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# Lecture 4: Total Probability Theorem and Bayes' Rule

# Total Probability Theorem

- Let $A_1, A_2, \dots, A_n$ form a partition of $\Omega$ and $P(A_i) > 0$
- Then, for any event $B$, we have

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B) \\ &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n). \end{aligned}$$

- This can be graphically explained as...

# Total Probability Theorem

In a certain assembly plant, three machines, $B_1$, $B_2$, and $B_3$, make 30%, 45%, and 25% of the products, respectively. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Answer is 0.0245.

# Bayes' Rule

Let $A_1, A_2, \ldots, A_n$ partition $\Omega$ and $P(A_i) > 0$. For any $B$ such that $P(B) > 0$,

$$
\begin{aligned}
P(A_i|B) &= \frac{P(A_i)P(B|A_i)}{P(B)} \\
&= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}
\end{aligned}
$$

# Example: Taking the Bus

Every morning you leave your house and take the first bus that goes to the university. There's a 25% chance that the first bus that comes will be a red bus and a 75% chance it will be a blue. If you take the red bus, you get to class late 20% of the time. If you take the blue bus, you get to class late 55% of the time. What's the probability that you get to class late?

- **Question:** What events are specified in the problem?
- **Answer:** $B_{red}$="red bus is first", $B_{blue}$="blue bus is first", $L$="get to class late"
- **Question:** What probabilities are specified in the problem?
- **Answer:** $P(B_{red}) = 0.25$ , $P(B_{blue}) = 0.75$ , $P(L|B_{red}) = 0.2$ , $P(L|B_{blue}) = 0.55$ .
- Need to compute $P(L)$: Since $B_{blue}$ and $B_{red}$ partition $\Omega$:

$$P(L) = P(L|B_{blue})P(B_{blue}) + P(L|B_{red})P(B_{red}) = 0.4625$$

# Example: Taking the Bus 2

Suppose the lecturer observes that you are late. What's the probability you caught the blue bus?
As before,

$$P(B_{red}) = 0.25 \; , \quad P(B_{blue}) = 0.75$$

$$P(L|B_{red}) = 0.2 \; , \quad P(L|B_{blue}) = 0.55 \; .$$

- Need to compute $P(B_{blue}|L)$:

$$P(B_{blue}|L) = \frac{P(B_{blue} \cap L)}{P(L)} = \frac{P(L|B_{blue})P(B_{blue})}{P(L)} = 0.891891\ldots$$

- First question uses the Total Probability Theorem and the question uses the Bayes' Rule.

# Example: Medical Testing and Diagnosis

Suppose there is a deadly disease that affects 1 in $10,000$ people. There is a lab test that can correctly identify positive cases 99% of the time and correctly identify negative cases 95% of the time. If you apply the test to a randomly selected individual, what is the probability that they will test positive?

- **Events:** $D =$"Have the disease" and $T =$"Test positive".
- **Relationships:** $D$ and $D^C$ partition $\Omega$.
- **Probabilities:** $P(D) = 0.0001$, $P(T|D) = 0.99$, $P(T^C|D^C) = 0.95$.
- **Question:** What is $P(T)$?
- **Answer:**

$$\begin{aligned} P(T) &= P(T|D)P(D) + P(T|D^C)P(D^C) \\ &= 0.99 \cdot 0.0001 + 0.05 \cdot 0.9999 = 0.0501 \ . \end{aligned}$$

# Example: Medical Testing and Diagnosis

Suppose there is a deadly disease that affects 1 in $10,000$ people. There is a lab test that can correctly identify positive cases 99% of the time and correctly identify negative cases 95% of the time. If a person's test result is positive, what is the probability that he/she has the disease?

- **Events:** $D =$ "Have the disease" and $T =$ "Test positive".
- **Relationships:** $D$ and $D^C$ partition $\Omega$.
- **Probabilities:** $P(D) = 0.0001$, $P(T|D) = 0.99$, $P(T^C|D^C) = 0.95$.
- **Question:** What is $P(D|T)$?

$$
\begin{aligned}
P(D|T) &= \frac{P(T|D)P(D)}{P(T)} \\
&= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^C)P(D^C)} \\
&= \frac{0.99 \cdot 0.0001}{0.99 \cdot 0.0001 + 0.05 \cdot 0.9999} = \frac{0.000099}{0.0501} = 0.001976
\end{aligned}
$$