COMPSCI 240: Reasoning Under Uncertainty

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Spring 2019

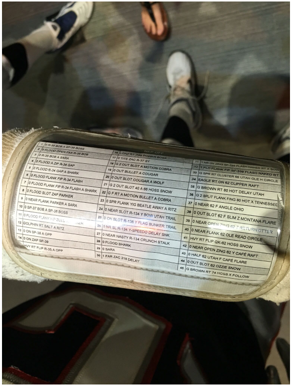
Lecture 3: Conditional Probability

Recap: Probability

- Sample space Ω: all possible outcomes of an experiment
- Probability of an event A (a set of possible outcomes) denoted as $P(A) \in [0, 1], \quad A \subset \Omega$
- Axioms of probability
  - Nonnegativity: $P(A) \geq 0$
  - Additivity: For any disjoint events $A_1, A_2, A_3, \ldots,$ $P(\cup_i A_i) = \sum_i P(A_i)$
  - Normalization: $P(\Omega) = 1$
- Conditional probabilities $P(A|B)$: probability of event A given event B happens
- New probability space:

$$P(A|B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B} = \frac{P(A \cap B)}{P(B)}.$$

“Real-world” example of conditional probability



New probability space $P(\cdot|B)$

Verify that the axioms of probability are satisfied!

- **Nonnegativity**: $P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$ since $P(A \cap B) \geq 0$
- **Additivity**: For any two disjoint sets A and C, show that $P(A \cup C|B) = P(A|B) + P(C|B)$.

$$
\begin{aligned}
P(A \cup C|B) &= \frac{P((A \cup C) \cap B)}{P(B)} \\
&= \frac{P((A \cap B) \cup (C \cap B)))}{P(B)} = \frac{P(A \cap B) + P(C \cap B)}{P(B)} \\
&= P(A|B) + P(C|B).
\end{aligned}
$$

- **Normalization**: New sample space is B.

$$
P(B|B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1
$$

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Example

Let us have two unfair coin tosses where the joint probability has

$$P(\{HH\}) = 1/2$$

$$P(\{HT\}) = 1/4$$

$$P(\{TH\}) = 1/8$$

$$P(\{TT\}) = 1/8$$

What is the probability that we have exactly one H **given that** the second toss shows H?

Define A and B first.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{TH\})}{P(\{HH, TH\})} = \frac{1/8}{1/2 + 1/8} = \frac{1}{5}.$$

Challenging Exercise

Example: Throw of two dice. Each of the 36 outcomes are equally likely

- $A =$ max of two dice is less than 5
- $B =$ min of the two dice is greater than 1

What is $P(A|B)$? (Please try this at home)

- $P(A) = \frac{16}{36}$
- $P(B) = \frac{25}{36}$
- $P(A \cap B) = \frac{9}{36}$
- $P(A|B) = \frac{9}{25}$

Sequential Model for Conditional Probabilities

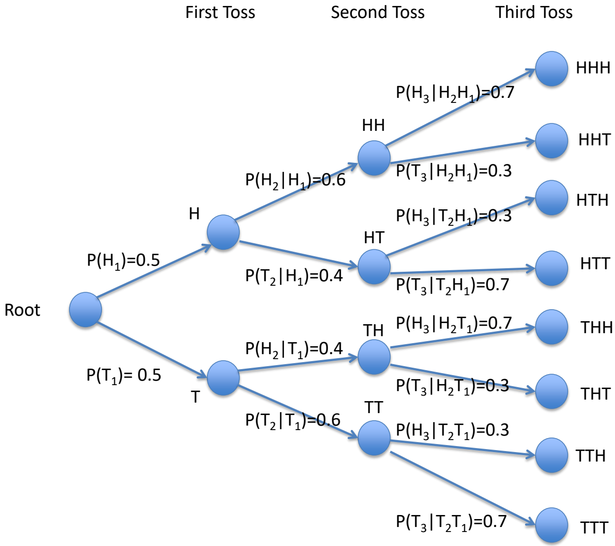
Many experiments have a sequential characteristic: the future outcomes depending on the past.

For example, consider an example involving three coin tosses.

- The first toss is unbiased (fair): $P(H) = 0.5$ and $P(T) = 0.5$.
- Based on the outcome of the first toss, the second toss is biased towards that outcome by 60%.
  - For example, if the outcome of the first toss is H, then the second toss has $P(H) = 0.6$ and $P(T) = 0.4$.
- Based on the outcome of the second toss, the third toss is biased towards that outcome by 70%.

Let us draw a **tree-based sequential description**.

Sequential Model for Conditional Probabilities

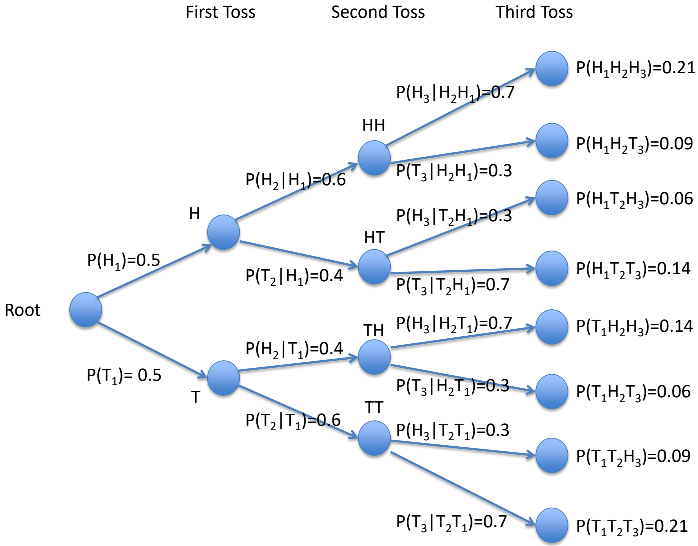


Sequential Model for Conditional Probabilities

How to setup a tree-based sequential description and use it?

1. Leaves represent events of interest, which occur in a sequential manner
2. Branches represent the conditional probability
3. The probability of the end-leaf can be computed by multiplying conditional probabilities from the root.

Sequential Model for Conditional Probabilities



Multiplication Rule

- We learned conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

which can be re-written as

$$P(A \cap B) = P(B)P(A|B) = P(A)(B|A)$$

- Now, what about

$$\begin{aligned} P(A \cap B \cap C) &= P((A \cap B) \cap C) \\ &= P(D \cap C), \text{ where } D = (A \cap B) \\ &= P(D)P(C|D) \\ &= P(A \cap B)P(C|A \cap B) \\ &= P(A)P(B|A)P(C|A \cap B) \end{aligned}$$

These are other equivalent results for $P(A \cap B \cap C)$.

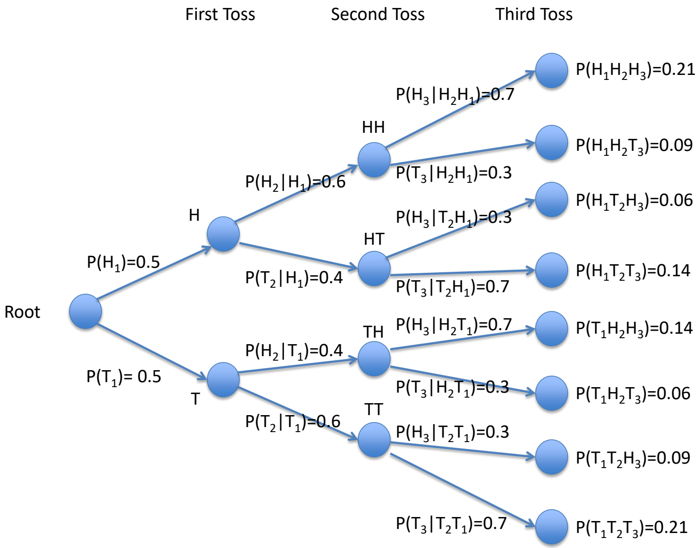
Multiplication Rule

In general,

$$\begin{aligned} P(\cap_{i=1}^{n} A_i) &\equiv P(A_1 \cap A_2 \cap \ldots A_n) \\ &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_n| \cap_{i=1}^{n-1} A_i) \end{aligned}$$

Sequential Model

Probability of a leaf of this tree?



More exercise

Three cards are drawn from an ordinary 52-card deck without replacement (drawn cards are not placed back in the deck). We wish to find the probability that none of the three cards is a heart. We assume that at each step, each one of the remaining cards is equally likely to be picked. (see, Textbook)

More exercise

- $A_i =$ first draw is not a heart, $i = 1, 2, 3$
- $P(A_1) = \frac{39}{52}$
- $P(A_2|A_1) = \frac{38}{51}$.
- $P(A_3|A_2 \cap A_1) = \frac{37}{50}$
- $P(A_1 \cap A_2 \cap A_3) = \frac{39 \cdot 38 \cdot 37}{52 \cdot 51 \cdot 50}$

The sequential model