# COMPSCI 240: Reasoning Under Uncertainty

Nic Herndon and Andrew Lan

University of Massachusetts at Amherst

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# Lecture 29: Bayesian Networks

# Review

- In practice, it is much common to encounter real-world problems that involve measuring multiple random variables $X_1, ..., X_n$ for each repetition of the experiment, with RVs that may have complex relationships among themselves.

- Chain Rule for $n$ random variables

$$P(X_n, \cdots, X_1) = P(X_n|X_{n-1}, \cdots, X_1)P(X_{n-1}, \cdots, X_1)$$

- Marginal probabilities for multiple discrete random variables $X_1, \cdots X_n$ with joint PMF, denoted as $P(X_1, \cdots X_n)$, could be computed as

$$P(X_1 = x_1) = \sum_{x_2} \cdots \sum_{x_n} P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$$

# The Curse of Dimensionality

- Suppose we have an experiment where we obtain the values of $d$ random variables $X_1, ..., X_d$, where each variable has binary outcomes (for simplicity).
- **Question:** How many **numbers** does it take to write down a joint distribution for them?
- **Answer:** The number of d-bit sequences is $2^d$. Because we know that the probabilities have to add up to 1, we need to write down $\mathbf{2^d - 1}$ numbers to specify the full joint PMF on $d$ binary variables.

# How Fast is Exponential Growth?

- $2^d - 1$ grows **exponentially** as $d$ increases **linearly**:

| $d$ | $2^d - 1$ |
|---|---|
| 1 | 1 |
| 10 | 1023 |
| 100 | 1,267,650,600,228,229,401,496,703,205,375 |
| ⋮ | ⋮ |

- Storing the full joint PMF for 100 binary variables would take about $10^{30}$ real numbers or about $10^{18}$ **terabytes** of storage!
- Joint PMFs grow in size so rapidly, we have no hope whatsoever of storing them explicitly for problems with more than about 30 (binary) random variables.

# Factorizing Joint Distributions

- To address this, we start by *factorizing the joint distribution*, i.e., re-writing the joint distribution as a product of conditional PMFs over single variables (called factors).
- If we know some conditional independency between the variables, we can save some space.
- Keeping track of all the conditional independence assumptions gets tedious when there are a lot of variables.
- To get around this problem, we use "Bayesian Networks" to express the conditional independence structure of these models.
  - A Bayesian network uses conditional independence assumptions to more compactly represent a joint PMF of many random variables.

# Bayesian Networks

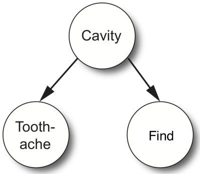
- We use a Directed Acyclic Graph (DAG) to encode conditional independence assumptions.
  - Nodes $X_i$ in the graph $G$ represent random variables.
  - A directed edge $X_j \rightarrow X_i$ means $X_i$ directly depends on $X_j$ (not causation!).
  - We also define that $X_j$ is a "parent" of $X_i$.
  - The set of variables that are parents of $X_i$ is denoted $Pa_i$.
  - $X_i$ is independent of all its nondescendants given $Pa_i$.
  - The factor associated with variable $X_i$ is $P(X_i|Pa_i)$.

# Example: Bayesian Network

- Toothache: boolean variable indicating whether the patient has a toothache
- Cavity: boolean variable indicating whether the patient has a cavity
- Catch/Find: whether the dentist's probe catches in the cavity
- We had

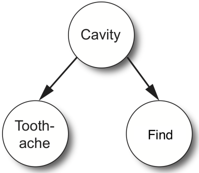
$$P(Find|Toothache, Cavity) = P(Find|Cavity)$$
$$P(Toothache|Find, Cavity) = P(Toothache|Cavity)$$

- This can be graphically represented as



# Example: Bayesian Network

- Given a BayesNet (DAG),




- Thus,

$$\begin{aligned} P(C, T, F) &= P(F|T, C)P(T|C)P(C) \\ &= P(F|C)P(T|C)P(C) \end{aligned}$$

$$\begin{aligned} P(C, T, F) &= P(T|F, C)P(F|C)P(C) \\ &= P(T|C)P(F|C)P(C) \end{aligned}$$

# Bayesian Networks vs. Markov Chains

- Do not confuse the *Bayesian Networks* and the *Transition Probability Graphs of Markov Chains*.
- These two graphs look similar (both have circles with arrows) but represent two vastly different entities.
- In Transition Probability Graphs, **nodes** represent all possible **states**, and **arrows** represents the **probability of transition** from one state to another (with numbers written on it).
- In Bayesian Networks, **nodes** represent all possible **random variables**, and **arrows** represents **dependencies** between the random variables (no numbers associated with it).

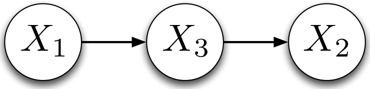
# The Bayesian Network Theorem

- **Definition:** A joint PMF $P(X_1, ..., X_d)$ is a Bayesian network with respect to a directed acyclic graph $G$ with parent sets $\{Pa_1, ..., Pa_d\}$ if and only if:

$$P(X_1, ..., X_d) = \prod_{i=1}^{d} P(X_i | Pa_i)$$

- In other words, to be a valid Bayesian network for a given graph $G$, the joint PMF must factorize according to $G$.

# 3 Cases of Conditional Independence to Remember

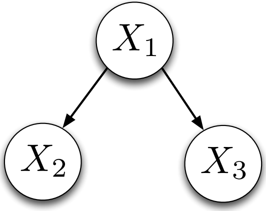



$$Pa_1 = \{\}, Pa_3 = \{X_1\}, Pa_2 = \{X_3\}$$

$$P(X_1 = a_1, X_2 = a_2, X_3 = a_3) =$$

$$P(X_1 = a_1)P(X_3 = a_3 | X_1 = a_1)P(X_2 = a_2 | X_3 = a_3)$$

# 3 Cases of Conditional Independence to Remember



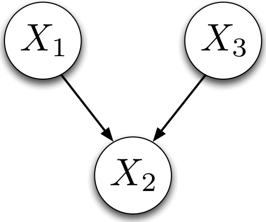

$$Pa_1 = \{\}, Pa_3 = \{X_1\}, Pa_2 = \{X_1\}$$

$$P(X_1 = a_1, X_2 = a_2, X_3 = a_3) =$$

$$P(X_1 = a_1)P(X_3 = a_3|X_1 = a_1)P(X_2 = a_2|X_1 = a_1)$$

- Note that $X_2$ and $X_3$ are conditionally independent given $X_1$.

# 3 Cases of Conditional Independence to Remember

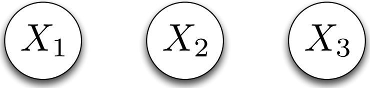


$$Pa_1 = \{\}, Pa_3 = \{\}, Pa_2 = \{X_1, X_3\}$$

$$P(X_1, X_2, X_3) = P(X_1)P(X_3)P(X_2|X_1, X_3)$$

- Note that $X_1$ is not independent of $X_3$ given $X_2$.

# If All Nodes Are Independent



$$Pa_1 = \{\}, Pa_3 = \{\}, Pa_2 = \{\}$$

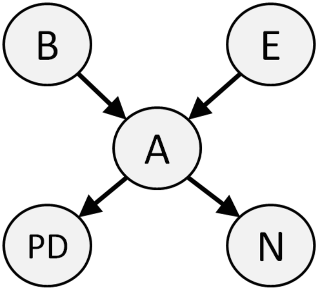
$$P(X_1, X_2, X_3) = P(X_1)P(X_3)P(X_2)$$

# The Alarm Network: Random Variable

- Consider the following situation:
- You live in quiet neighborhood in the suburbs of LA. There are two reasons the alarm system in your house will go off: your house is broken into or there is an earthquake. If your alarm goes off you might get a call from the police department. You might also get a call from your neighbor.
- **Question** What random variables can we use to describe this problem?
- **Answer:** Break-in (B), Earthquake (E), Alarm (A), Police Department calls (PD), Neighbor calls (N).

# The Alarm Network: Factorization

- **Question** What direct dependencies might exist between the random variables $B, E, A, PD, N$?




- **Question:** What is the factorization implied by the graph?
- $P(B, E, A, PD, N) = P(B)P(E)P(A|B, E)P(PD|A)P(N|A)$

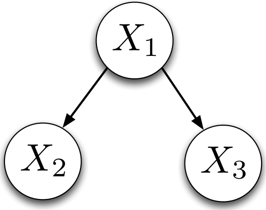
# From Graphs to Factorizations and Back

- If we have a valid graph, we can infer the parent sets and the factors.
- If we have a valid set of factors, we can infer the parent sets and the graph.
- If we have a "text" that describes a problem, we can infer a graph and set of factors.

# Example: Factorization to Graph

$$P(X_1, X_2, X_3) = P(X_1)P(X_3|X_1)P(X_2|X_1)$$

$$Pa_1 = \{\}, Pa_3 = \{X_1\}, Pa_2 = \{X_1\}$$

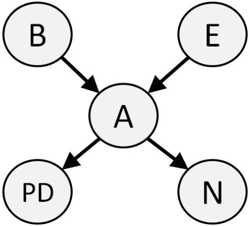


# The Alarm Network: Factor Tables

$$P(B, E, A, PD, N) = P(B)P(E)P(A|B, E)P(PD|A)P(N|A)$$

| P(B=1) |
|---|
| 0.001 |

| P(E=1) |
|---|
| 0.002 |




| B | E | P(A=1\|B,E) |
|---|---|---|
| 1 | 1 | 0.950 |
| 1 | 0 | 0.940 |
| 0 | 1 | 0.290 |
| 0 | 0 | 0.001 |

| A | P(PD=1\|A) |
|---|---|
| 1 | 0.900 |
| 0 | 0.005 |

| A | P(N=1\|A) |
|---|---|
| 1 | 0.750 |
| 0 | 0.100 |

# The Alarm Network: Joint Query

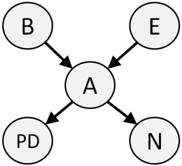
- **Question:** What is the probability that there is a break-in, but no earthquake, the alarm goes off, the police call, but your neighbor does not call?

$$P(B=1, E=0, A=1, PD=1, N=0)$$
$$= P(B=1)P(E=0)P(A=1|B=1,E=0)P(PD=1|A=1)P(N=0|A=1)$$
$$= 0.001 \cdot (1-0.002) \cdot 0.94 \cdot 0.9 \cdot (1-0.75) = 0.00021...$$

| P(B=1) |
|---|
| 0.001 |

| P(E=1) |
|---|
| 0.002 |

| B | E | P(A=1\|B,E) |
|---|---|---|
| 1 | 1 | 0.950 |
| 1 | 0 | 0.940 |
| 0 | 1 | 0.290 |
| 0 | 0 | 0.001 |




| A | P(PD=1\|A) |
|---|---|
| 1 | 0.900 |
| 0 | 0.005 |

| A | P(N=1\|A) |
|---|---|
| 1 | 0.750 |
| 0 | 0.100 |