Lecture 26: Markov Chains III
Recap: Discrete Markov Chain

- We consider discrete-time Markov chain, in which the state changes at certain discrete time instances, indexed by an integer variable $t$.
- A discrete Markov chain defines a series of random variables $X_t$, e.g., $\{X_0, X_1, X_2, \ldots \}$.
- A Markov Chain consists:
  - **State space**: a set of states in which the chain can be described at time $t$:
    $$S = \{s_1, \ldots, s_k\}$$
  - **Transition probabilities** that describe the probability of transitioning from a state at $t - 1$ to another state at $t$:
    $$P (X_t = s_j | X_{t-1} = s_i) = p_{ij} \text{ for all } 1 \leq i, j \leq k$$
  - An initial state $X_0$, in which the chain is initiated.
Markov Chain

• Write the **probability distribution** of each $X_t$ as

$$
\nu_t = \langle \nu_t[1], \nu_t[2], \ldots, \nu_t[k] \rangle \\
= \langle P (X_t = s_1), P (X_t = s_2), \ldots, P (X_t = s_k) \rangle.
$$

• If we happen to know the $\nu_t$, then we can compute $\nu_{t+1}$ using the **Total Probability Law**.

$$
P (X_{t+1} = s_j) = \sum_i P (X_{t+1} = s_j | X_t = s_i) P (X_t = s_i).
$$

or

$$
\nu_{t+1}[j] = \sum_i \nu_t[i] p_{ij}
$$
Steady State Distribution

Do all Markov chains have the property that eventually the distribution settles to the “same steady” state regardless of the initial state?

**Definition**

We have

\[ v = \lim_{t \to \infty} v_t \]

\[ \langle v[1], v[2], \ldots, v[k] \rangle = \lim_{t \to \infty} \langle v_t[1], v_t[2], \ldots, v_t[k] \rangle \]

If we have

\[ v[j] = \sum_{i=1}^{k} p_{ij} v[i] \text{ for } j = 1, \ldots, k \]

and

\[ \sum_{j=1}^{k} v[j] = 1 \]

Then, we say \( v \) is a **steady state distribution** for the Markov Chain.
Classification of States

- We say that a state $i$ is **recurrent** if for every $j$ that is accessible from $i$, $i$ is also accessible from $j$.
  
  - Denoting $A(i)$ as a set of states that are accessible from $i$, for all $j$ that belong to $A(i)$ we have that $i$ belongs to $A(j)$.

- If $i$ is a recurrent state, the set of states $A(i)$ that are accessible from $i$ form a **recurrent class**.
  
  - States in $A(i)$ are all accessible from each other, and no state outside $A(i)$ is accessible from them.

- A state is called **transient** if it is not recurrent.

- A Markov chain with multiple recurrent classes **does not** converges to a unique steady state.
Classification of States

1 — 2 — 3 — 4
Classification of States

1 (Recurrent) → 2 (Transient) → 3 (Recurrent) → 4 (Recurrent)
Classification of States

Recurrent Class #1

Recurrent

Transient

Recurrent Class #2

Recurrent

Recurrent
Example

Consider the Markov Chain with transition matrix:

\[ A = \begin{pmatrix}
0 & 0.9 & 0.05 & 0.05 \\
0.2 & 0.8 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]
Recurrent States

- **Question**: Which of the following Markov chains have a single recurrent class?
Periodic Recurrent Class

Definition

- Consider a recurrent class.
- Let us group all the states into \( d \) disjoint groups of states \( S_1, \cdots, S_d \); a group has to contain at least one state.
- Such a recurrent class is called **periodic** if there exists at least one group (of states) in the chain that is visited with a period of \( T \). That is group(s) are visited at time \( \{T, 2T, 3T, 4T, \ldots\} \) steps for \( T \in \{2, 3, \ldots\} \).
- If a recurrent class is not periodic, we call the class **aperiodic**.
• **Question**: Which of the following Markov chains contain a single periodic recurrent class?
Periodic/Aperiodic Class

- **Question**: Does the following Markov chain contain a single periodic recurrent class?
Steady-State Convergence Theorem

Theorem
Consider a Markov chain with a single, aperiodic recurrent class. Then, the states in such a Markov chain has steady-state distribution.
Steady State Distribution Example

- Consider a Markov chain $C$ with 2 states and transition matrix

$$A = \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}$$

for some $0 < a, b < 1$

- Does $C$ has a single recurrent class?
- Is $C$ periodic?
- Then, what is its steady state distribution $\mathbf{v}$?
Example: Web Graph Transition Diagram
Markov Chains: Steady State Applications

- One of the most profitable applications of steady state theory in Markov chains is Google’s PageRank Algorithm.
- The states are all possible pages on the web.
- The probability of transitioning from page $i$ to page $j$ is the fraction of out-going links from page $i$ that point to page $j$.
- The steady state distribution tells you the proportion of time someone randomly surfing the web would end up on each page.
- Use this to rank among the pages that match a keyword search.