COMPSCI 240: Reasoning Under Uncertainty

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Lecture 26: Markov Chains III

Recap: Discrete Markov Chain

- We consider discrete-time Markov chain, in which the state changes at certain discrete time instances, indexed by an integer variable t.
- A discrete Markov chain defines a series of random variables X_t, e.g., $\{X_0, X_1, X_2, \ldots\}$.
- A Markov Chain consists:
  - **State space**: a set of states in which the chain can be described at time t:

$$\mathcal{S} = \{s_1, \ldots, s_k\}$$

  - **Transition probabilities** that describe the probability of transitioning from a state at $t-1$ to another state at t:

$$P\left(X_t = s_j | X_{t-1} = s_i\right) = p_{ij} \text{ for all } 1 \leq i, j \leq k$$

  - An initial state X_0, in which the chain is initiated.

Markov Chain

- Write the **probability distribution** of each X_t as

$$\begin{aligned} v_t &= \langle v_t[1], v_t[2], \ldots, v_t[k] \rangle \\ &= \langle P(X_t = s_1), P(X_t = s_2), \ldots, P(X_t = s_k) \rangle. \end{aligned}$$

- If we happen to know the v_t, then we can compute v_{t+1} using the **Total Probability Law**.

$$P(X_{t+1} = s_j) = \sum_i P(X_{t+1} = s_j | X_t = s_i) P(X_t = s_i).$$

or

$$v_{t+1}[j] = \sum_i v_t[i] p_{ij}$$

Steady State Distribution

Do all Markov chains have the property that eventually the distribution settles to the "same steady" state regardless of the initial state?

Definition

We have

$$v = \lim_{t \to \infty} v_t$$
$$\langle v[1], v[2], \ldots, v[k] \rangle = \lim_{t \to \infty} \langle v_t[1], v_t[2], \ldots, v_t[k] \rangle$$

If we have

$$v[j] = \sum_{i=1}^{k} p_{ij} v[i] \text{ for } j = 1, \cdots, k$$

and

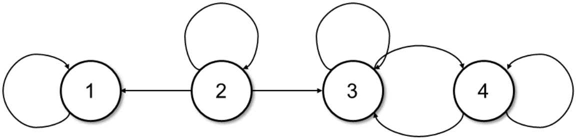
$$\sum_{j=1}^{k} v[j] = 1$$

Then, we say v is a **steady state distribution for the Markov Chain.**

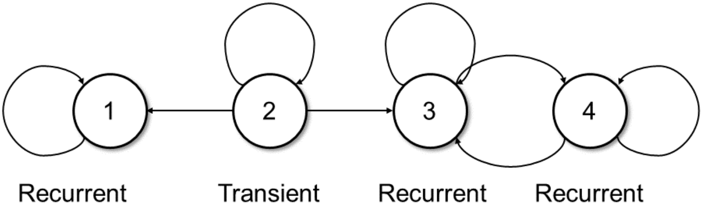
Classification of States

- We say that a state i is **recurrent** if for every j that is accessible from i, i is also accessible from j.
  - Denoting $A(i)$ as a set of states that are accessible from i, for all j that belong to $A(i)$ we have that i belongs to $A(j)$.
- If i is a recurrent state, the set of states $A(i)$ that are accessible from i form a **recurrent class**.
  - States in $A(i)$ are all accessible from each other, and no state outside $A(i)$ is accessible from them.
- A state is called **transient** if it is not recurrent.
- A Markov chain with multiple recurrent classes **does not** converges to a unique steady state.

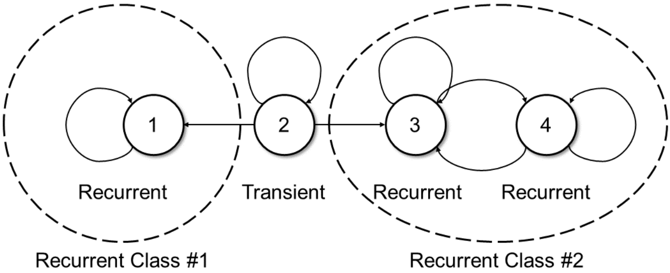
Classification of States



Classification of States



Classification of States



Classification of States

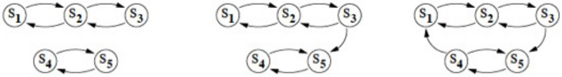
Example

Consider the Markov Chain with transition matrix:

$$A = \begin{pmatrix} 0 & 0.9 & 0.05 & 0.05 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Recurrent States

- **Question**: Which of the following Markov chains have a single recurrent class?



- **Answer**: Right two chains

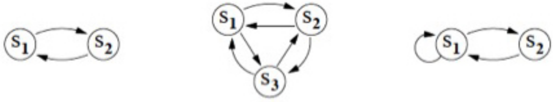
Periodic Recurrent Class

Definition

- Consider a recurrent class.
- Let us group all the states into d disjoint groups of states $S_1, \cdots, S_d$; a group has to contain at least one state.
- Such a recurrent class is called **periodic** if there exists at least one group (of states) in the chain that is visited with a period of T. That is, group(s) are visited at time $\{T, 2T, 3T, 4T, \ldots\}$ steps for $T \in \{2, 3, \ldots\}$.
- If a recurrent class is not periodic, we call the class **aperiodic**.

Periodic/Aperiodic Class

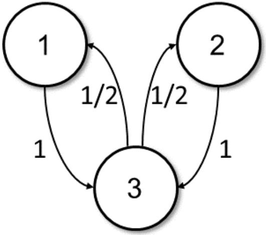
- **Question**: Which of the following Markov chains contain a single periodic recurrent class?



- **Answer**: Only the one to the left (with period of 2).

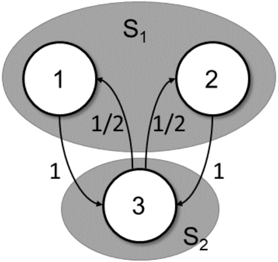
Periodic/Aperiodic Class

- **Question**: Does the following Markov chain contain a single periodic recurrent class?



Periodic/Aperiodic Class

- **Question**: Does the following Markov chains contain a single periodic recurrent class?



- **Answer**: Yes. Group 1 (State 1 and 2) and Group 2 (States 3) occur periodically.

Steady-State Convergence Theorem

Theorem

Consider a Markov chain with a single, aperiodic recurrent class. Then, the states in such a Markov chain have steady-state distribution.

Steady State Distribution Example

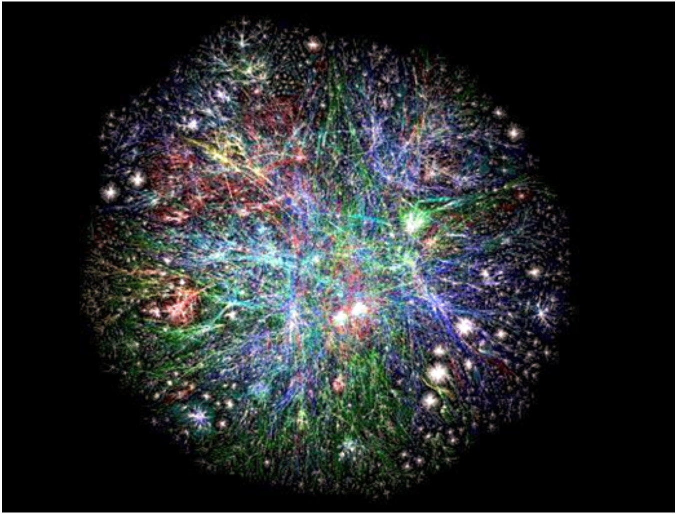
- Consider a Markov chain C with 2 states and transition matrix

$$A = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

  for some $0 < a, b < 1$
- Does C have a single recurrent class? **Yes.**
- Is C periodic? **No**, as long as $0 < a, b < 1$
- Then, what is its steady state distribution $\mathbf{v}$?
- Let $\mathbf{v} = (c, 1-c)$ be a steady state distribution.
- Solving $v[j] = \sum_{k=1}^{m} v[k] p_{kj}$ for gives:

$$v^* = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$$

Example: Web Graph Transition Diagram



Markov Chains: Steady State Applications

- One of the most profitable applications of steady state theory in Markov chains is Google's PageRank Algorithm.
- The states are all possible pages on the web.
- The probability of transitioning from page i to page j is the fraction of out-going links from page i that point to page j.
- The steady state distribution tells you the proportion of time someone randomly surfing the web would end up on each page.
- Use this to rank among the pages that match a keyword search.