Lecture 23: Game Theory III
Zero-Sum Games

Definition
A two-player zero-sum game consists of a set of actions $A_i$ for Player $P_i$ and $A_j$ for Player $P_j$, where each strategy profile $a \in A_i \times A_j$ has the payoff function $u_1(a) + u_2(a) = 0$.

For two-finger Morra, the payoff matrix is

<table>
<thead>
<tr>
<th></th>
<th>1 B Finger</th>
<th>2 B Finger</th>
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<tbody>
<tr>
<td>1 A Finger</td>
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<td>−3, +3</td>
<td>+4, −4</td>
</tr>
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</table>

This game is a zero-sum game.
Pure Strategies vs. Mixed Strategies

- **Pure Strategy**: Players choose a strategy to select a single action and play it - so far we have considered this scenario.

- **Mixed Strategy**: Players randomize over the set of available actions according to some probability distribution - a player randomizes and mixes between different actions.
# Zero-Sum Games

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And there exists no clear Nash Equilibrium when we consider pure strategies.

But, remember that

**Theorem (Nash)**

*Every game where each player has a finite number of options, has at least one Nash equilibrium.*

If no equilibrium exists in pure strategies, one must exist in **mixed strategies**.
Hawks and Doves

- Previous discussed hawks and doves example with the following payoff matrix have both the 1) pure strategy and 2) mixed strategy Nash equilibrium

<table>
<thead>
<tr>
<th>B is a Hawk</th>
<th>B is a Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>A is a Hawk</td>
<td>-25, -25</td>
</tr>
<tr>
<td>A is a Dove</td>
<td>0, 50</td>
</tr>
</tbody>
</table>

- A plays hawk and B plays Dove (or vice versa) is a pure-strategy Nash equilibrium.
- A and B play hawks with \( p = q = \frac{7}{12} \) is a mixed-strategy Nash equilibrium.
- The three Nash equilibria can be summarized as
  - \( p = 0 \) and \( q = 1 \) (Pure Strategy)
  - \( p = 1 \) and \( q = 0 \) (Pure Strategy)
  - \( p = \frac{7}{12} \) and \( q = \frac{7}{12} \) (Mixed Strategy)
Example

Consider the following payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
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<tr>
<td>Up</td>
<td>+3, -3</td>
<td>-2, +2</td>
</tr>
<tr>
<td>Down</td>
<td>-1, +1</td>
<td>0, 0</td>
</tr>
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- Is there a pure-strategy Nash equilibrium?
- Is this a zero-sum game?
- Find a mixed-strategy Nash equilibrium?
Example: Three-finger Morra

- Alice and Bob play a game
- Simultaneously Alice picks $a \in \{1, 2, 3\}$ and Bob picks $b \in \{1, 2, 3\}$
- Bob pays Alice $(a + b)$ if $a + b$ is even
- Alice pays Bob $(a + b)$ if $a + b$ is odd
- The payoff matrix is

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<td>−5, +5</td>
</tr>
<tr>
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<td>+4, −4</td>
<td>−5, +5</td>
<td>+6, −6</td>
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Analysis of Three-finger Morra (1/2)

- Suppose B plays “1” with probability $r$, “2” with probability $s$, and “3” with probability $1 - r - s$
- If A plays “1” then A’s expected reward is
  \[2r - 3s + 4(1 - r - s) = 4 - 2r - 7s\]
- If A plays “2” then A’s expected reward is
  \[-3r + 4s - 5(1 - r - s) = -5 + 2r + 9s\]
- If A plays “3” then A’s expected reward is
  \[4r - 5s + 6(1 - gr - s) = 6 - 2r - 11s\]
- Hence, for $r = 1/4, s = 1/2$, A gets expected return of 0
Analysis of Three-finger Morra (2/2)

- Suppose A plays “1” with probability $t$, “2” with probability $u$, and “3” with probability $1 - t - u$
- If B plays “1” then B’s expected reward is
  \[ -2t + 3u - 4(1 - t - u) = -4 + 2t + 7u \]
- If B plays “2” then B’s expected reward is
  \[ 3t - 4u + 5(1 - t - u) = 5 - 2t - 9u \]
- If B plays “3” then B’s expected reward is
  \[ -4t + 5u - 6(1 - t - u) = -6 + 2t + 11u \]
- Hence, for $t = 1/4$, $u = 1/2$, B gets expected return of 0

In sum, both players show one finger with prob. 1/4 and two fingers with prob. 1/2 is a mixed-strategy Nash equilibrium.
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We computed the mixed-strategy Nash equilibrium when \( p = q = 7/12 \).

**Question**: In a zero-sum game, is a Nash equilibrium the strategy that maximizes the players reward?
Analysis of Two-finger Morra

- Suppose Bob randomizes his action by playing “1” with probability $q$ and “2” with probability $1 - q$

  $$P(B = 1) = q \text{ and } P(B = 2) = 1 - q.$$ 

- If Alice plays “1” then Alice has expected payoff

  $$2q - 3(1 - q) = 5q - 3$$

- If Alice plays “2” then Alice has expected payoff

  $$-3q + 4(1 - q) = 4 - 7q$$

- Then, how should Alice play to maximize her expected reward if she knows the value of $q$?

- She will choose a pure strategy that will yield the better reward.

  $$\max(5q - 3, 4 - 7q)$$
Analysis of Two-finger Morra

Maximum Payoff for Alice

\[ \max(5q-3, 4-7q) \]

Maximum Payoff for Bob

\[ \max(3-5q, 7q-4) \]