COMPSCI 240: Reasoning Under Uncertainty

Nic Herndon and Andrew Lan

University of Massachusetts at Amherst

Spring 2019

Lecture 23: Game Theory III

Zero-Sum Games

Definition

A *two-player zero-sum game* consists of a set of actions A_i for Player P_i and A_j for Player P_j, where each strategy profile $a \in A_i \times A_j$ has the payoff function $u_1(a) + u_2(a) = 0$.

For two-finger Morra, the payoff matrix is

	1 B Finger	2 B Finger
1 A Finger	$+2, -2$	$-3, +3$
2 A Finger	$-3, +3$	$+4, -4$

This game is a zero-sum game.

Pure Strategies vs. Mixed Strategies

- **Pure Strategy**: Players choose a strategy to select a single action and play it - so far we have considered this scenario.
- **Mixed Strategy**: Players randomize over the set of available actions according to some probability distribution - a player randomizes and mixes between different actions.

Zero-Sum Games

For two-finger Morra, the payoff matrix is

	1 B Finger	2 B Finger
1 A Finger	$+2, -2$	$-3, +3$
2 A Finger	$-3, +3$	$+4, -4$

And there exists no clear Nash Equilibrium when we consider pure strategies.
But, remember that

Theorem (Nash)

Every game where each player has a finite number of options, has at least one Nash equilibrium.

If no equilibrium exists in pure strategies, one must exist in **mixed strategies**.

Hawks and Doves

- Previous discussed hawks and doves example with the following payoff matrix have both the 1) pure strategy and 2) mixed strategy Nash equilibrium

	B is a Hawk	B is a Dove
A is a Hawk	$-25, -25$	$50, 0$
A is a Dove	$0, 50$	$15, 15$

- A plays hawk and B plays Dove (or vice versa) is a pure-strategy Nash equilibrium.
- A and B play hawks with $p = q = 7/12$ is a mixed-strategy Nash equilibrium.
- The three Nash equilibira can be summarized as
  - $p = 0$ and $q = 1$ (Pure Strategy)
  - $p = 1$ and $q = 0$ (Pure Strategy)
  - $p = 7/12$ and $q = 7/12$ (Mixed Strategy)

Example

Consider the following payoff matrix

	Left	Right
Up	$+3, -3$	$-2, +2$
Down	$-1, +1$	$0, 0$

- Is there a pure-strategy Nash equilibrium?
  - No.
- Is this a zero-sum game?
  - Yes.
- Find a mixed-strategy Nash equilibrium?
  - $p = 1/6$ and $q = 1/3$.

Example: Three-finger Morra

- Alice and Bob play a game
- Simultaneously Alice picks $a \in \{1, 2, 3\}$ and Bob picks $b \in \{1, 2, 3\}$
- Bob pays Alice $\$(a + b)$ if $a + b$ is even
- Alice pays Bob $\$(a + b)$ if $a + b$ is odd
- The payoff matrix is

	1 B Finger	2 B Finger	3 B Finger
1 A Finger	$+2, -2$	$-3, +3$	$+4, -4$
2 A Finger	$-3, +3$	$+4, -4$	$-5, +5$
3 A Finger	$+4, -4$	$-5, +5$	$+6, -6$

Analysis of Three-finger Morra (1/2)

- Suppose B plays "1" with probability r, "2" with probability s, and "3" with probability $1 - r - s$
- If A plays "1" then A's expected reward is

$$2r - 3s + 4(1 - r - s) = 4 - 2r - 7s$$

- If A plays "2" then A's expected reward is

$$-3r + 4s - 5(1 - r - s) = -5 + 2r + 9s$$

- If A plays "3" then A's expected reward is

$$4r - 5s + 6(1 - gr - s) = 6 - 2r - 11s$$

- Hence, for $r = 1/4, s = 1/2$, A gets expected return of 0

Analysis of Three-finger Morra (2/2)

- Suppose A plays "1" with probability t, "2" with probability u, and "3" with probability $1 - t - u$
- If B plays "1" then B's expected reward is

$$-2t + 3u - 4(1 - t - u) = -4 + 2t + 7u$$

- If B plays "2" then B's expected reward is

$$3t - 4u + 5(1 - t - u) = 5 - 2t - 9u$$

- If B plays "3" then B's expected reward is

$$-4t + 5u - 6(1 - t - u) = -6 + 2t + 11u$$

- Hence, for $t = 1/4, u = 1/2$, B gets expected return of 0

In sum, both players show one finger with prob. $1/4$ and two fingers with prob. $1/2$ is a mixed-strategy Nash equilibrium.

Zero-Sum Games

For two-finger Morra, the payoff matrix is

	1 B Finger	2 B Finger
1 A Finger	$+2, -2$	$-3, +3$
2 A Finger	$-3, +3$	$+4, -4$

We computed the mixed-strategy Nash equilibrium when $p = q = 7/12$.

Question: In a zero-sum game, is a Nash equilibrium the strategy that maximizes the players reward?

Analysis of Two-finger Morra

- Suppose Bob randomizes his action by playing "1" with probability q and "2" with probability $1 - q$

$$P(B = 1) = q \text{ and } P(B = 2) = 1 - q.$$

- If Alice plays "1" then Alice has expected payoff

$$2q - 3(1 - q) = 5q - 3$$

- If Alice plays "2" then Alice has expected payoff

$$-3q + 4(1 - q) = 4 - 7q$$

- Then, how should Alice play to maximize her expected reward if she knows the value of q?
- She will choose a pure strategy that will yield the better reward.

$$max(5q - 3, 4 - 7q)$$

Analysis of Two-finger Morra