COMPSCI 240: Reasoning Under Uncertainty

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Lecture 22: Game Theory II

Nash Equilibrium

Definition

A Nash Equilibrium is a set of strategies for *each player* where no change by one player alone can improve his/her outcome; each player has no incentive to change his/her rather stable strategy.

- We are considering Nash Equilibrium where players do not randomize between two or more strategies (called Pure Strategy Nash Equilibrium).
- We only care about alternating strategies in an individual level, not in a group level where everyone collectively change strategies toward a single strategy.
- A Nash Equilibrium is a law that no one would want to break even in the absence of an effective police force.
- There may exist multiple Nash Equilibria.

Theorem (Nash)

Every game where each player has a finite number of options, has at least one Nash equilibrium.

Iterated Elimination of Strictly Dominated Strategies vs. Nash Equilibrium

- Strictly dominated strategies cannot be a part of a Nash equilibrium.
- After completing the IESDS, if there exists only one strategy for each player remaining, that strategy set is the unique Nash equilibrium.
- Even if there exists no solutions from the IESDS, there may exist Nash Equilibria.

Stop Light Example

	Go	Stop
Go	$-5, -5$	$1, 0$
Stop	$0, 1$	$-1, -1$

- This has two Nash Equilibria
  - A = Go and B = Stop (e.g., when A has a green light)
  - A = Stop and B = Go (e.g., when B has a green light)
- No Nash Equilibrium where both players play the same pure strategy.

Hawks and Doves

- Two birds meet over a piece of food and have to decide whether to act aggressive (**hawkish**) or passive (**dovish**)

	B is a Hawk	B is a Dove
A is a Hawk	$-25, -25$	$50, 0$
A is a Dove	$0, 50$	$15, 15$

  - No Nash Equilibrium where both players play the same pure strategy.
  - If A and B are Hawks, both would prefer to switch to Doves
  - If A and B are Doves, both would prefer to switch to Hawks
- A plays Hawk and B plays Dove is a Nash equilibria and vice versa
  - If A knows that B plays Hawks (and that B will not change his strategy), she must play Doves
  - If B knows that A plays Hawks (and that B will not change his strategy), he must play Doves

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Pure Strategies vs. Mixed Strategies

- **Pure Strategy**: Players choose a strategy to select a single action and play it - so far we have considered this scenario.
- **Mixed Strategy**: Players randomize over the set of available actions according to some probability distribution - a player randomizes and mixes between different actions.

Example: Two-finger Morra

Let us assume that

- Alice and Bob play a game
- Simultaneously Alice picks an action $a \in \{1, 2\}$ and Bob picks an action $b \in \{1, 2\}$
- Bob pays Alice \$$(a + b)$ if $a + b$ is even
- Alice pays Bob \$$(a + b)$ if $a + b$ is odd

The payoff matrix looks like:

	1 B Finger	2 B Finger
1 A Finger	$+2, -2$	$-3, +3$
2 A Finger	$-3, +3$	$+4, -4$

Do we have a Nash equilibrium?
Obviously if Bob always plays the same number, Alice can take advantage of this.

Zero-Sum Games

Definition

A *Two-player zero-sum game* consists of a set of actions A_i for Player P_i and A_j for Player P_j, where each strategy profile $a \in A_i \times A_j$ has the payoff function $u_1(a) + u_2(a) = 0$.

For two-finger Morra, the payoff matrix is

	1 B Finger	2 B Finger
1 A Finger	$+2, -2$	$-3, +3$
2 A Finger	$-3, +3$	$+4, -4$

This game is a zero-sum game.

Zero-Sum Games

For two-finger Morra, the payoff matrix is

	1 B Finger	2 B Finger
1 A Finger	$+2, -2$	$-3, +3$
2 A Finger	$-3, +3$	$+4, -4$

And there exists no clear Nash Equilibrium when we consider pure strategies.
But, remember that

Theorem (Nash)

Every game where each player has a finite number of options, has at least one Nash equilibrium.

If no equilibrium exists in pure strategies, one must exist in **mixed strategies**.

Analysis of Two-finger Morra

- Suppose Bob randomizes his action by playing "1" with probability q and "2" with probability $1 - q$

$$P(B = 1) = q \text{ and } P(B = 2) = 1 - q.$$

- If Alice plays "1" then Alice has expected payoff

$$2q - 3(1 - q) = 5q - 3$$

- If Alice plays "2" then Alice has expected payoff

$$-3q + 4(1 - q) = 4 - 7q$$

- If Alice's payoffs are equal, then Alice does not have to prefer one action over the other, such that she does not expect to do better by changing her strategy (i.e., changing the value of p).
  - This meets the definition of a Nash equilibrium.
  - Do not confuse between the definitions of a strategy vs. an action.

Analysis of Two-finger Morra

- That is, when Bob's strategy makes the payoffs of Alice's actions equal

$$5q - 3 = 4 - 7q$$

or equivalently when

$$q = 7/12$$

- Then, Alice can choose also a strategy (since the payoffs are indifferent for her) to make Bob's actions to have the equal payoff.
- Similarly, that is when Alice's strategy is $p = 7/12$.
- This also means that Bob is completely satisfied for playing a mixed strategy with $q = 7/12$.
- Hence $p = q = 7/12$ is a mixed-strategy Nash equilibrium; players do not have any incentives to change their strategies.