Lecture 21: Game Theory I
Game Theory

- Our textbook does not include the materials related to Game Theory (this & next two lectures).
- We will use lecture slides for the next three lectures to provide a brief introduction to Game Theory.
Game Theory

- Game theory studies what happens when self-interested agents interact.
- Each player (or agent) has his/her own description of which states of the world he/she likes (or the states that benefits the agent the most).
- A player’s benefits can be represented using a payoff matrix that maps the states to real numbers.
- For now, let us assume that all players select a single action and play – all players have pure strategy.
Prisoner’s Dilemma

- Two suspects are being held pending trial for a crime they are alleged to have committed. The prosecutor does not have clear evidence who’s the criminal and needs one of suspects to rat out the other.

- Thus, the prosecutor offers each a deal: “Give evidence against your partner and you’ll go free, unless your partner also confesses. If both confess, both get 5 year sentences. If neither confess, both get 1 year sentences. If you don’t confess but your partner does, you get 10 years!”

- Prisoner $A$ and $B$ have two actions: 1) Confess or 2) Stay silent.

- Suppose the prisoners’ interests are in minimizing the number of years in jail.
Prisoner’s Dilemma

- The payoff matrix looks like the following:

<table>
<thead>
<tr>
<th></th>
<th>B Confesses</th>
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<tbody>
<tr>
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<td>0, −10</td>
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<td>−10, 0</td>
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</tr>
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In each entry, the first number is A’s payoff and the second number is B’s payoff.

- What would be best action that each prisoner can take without discussing together?
- Given that B confesses, then what action should A take?
- Given that B stays silent, then what action should A take?
- Similarly, given actions of A, what actions shall B take?
Strict Domination

- A strategy $s_i$ of a player $P_i$ strictly dominates another strategy $s'_i$ of the player, if $s_i$ generates a greater payoff than $s'_i$.
- More formally, let $P_i$ represent a player $i$ and $P_{-i}$ represent all other players but $i$. Furthermore, Let $s_i$ and $s'_i$ be two strategies of $P_i$, and $S_{-i}$ be the set of all strategies of the remaining players.
- Then, $s_i$ strictly dominates $s'_i$ if for all $s_{-i} \in S_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$, where $u_i$ is the payoff function.
Prisoner’s Dilemma

- Confessing strictly dominates all other strategies (stay silent).

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- We have four possible states in this game but there’s only one sensible outcome: i.e., both suspects confess.

- It is interesting because staying quiet produces an outcome that is mutually (and individually) better for both suspects.

- This is because each player (suspects) plays the game for their own interests.
Dominated Strategy

- A strategy $s_i$ of a player $P_i$ is **strictly dominated** if some (not all) other strategies $s'_i$ strictly dominates $s_i$. 
Iterated Elimination of Strictly Dominated Strategies (IESDS)

- Intuitively, all strategies that are strictly dominated by other strategies can be ignored, since they can never be best responses to any moves by the other players.
- Consider the following payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>13, 3, 1, 4</td>
<td>7, 3</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>4, 1, 3, 3</td>
<td>6, 2</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>-1, 9</td>
<td>2, 8</td>
<td>8, -1</td>
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- Strategy R of Player 2 is strictly dominated by C, so remove it.
Iterated Elimination of Strictly Dominated Strategies

• The payoff matrix reduced after removing the dominated strategy R.

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• Note that D is strictly dominated by M, so remove it.
Iterated Elimination of Strictly Dominated Strategies

- The pay-off matrix reduced to

$$
\begin{array}{c|cc}
 & L & C \\
\hline
U & 13, 3 & 1, 4 \\
M & 4, 1, 3 & 3, 3 \\
\end{array}
$$

- Now L is dominated by C, so remove it.
- Then, the only reasonable strategy to be played is M for Player 1 and C for Player 2.
Hawks and Doves

- Two birds meet over a piece of food and have to decide whether to act aggressive (hawkish) or passive (dovish)
  - If a hawk meets a dove, the hawk gets the food worth 50 points
  - If two hawks meet, they both loose -25 points
  - If two doves meet, they both get 15 points

- Can represent this as:

<table>
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<th>B is a Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>A is a Hawk</td>
<td>50, 0</td>
<td>-25, -25</td>
</tr>
<tr>
<td>A is a Dove</td>
<td>0, 50</td>
<td>15, 15</td>
</tr>
</tbody>
</table>

- We cannot solve this game by looking for strictly dominating strategies.
- We are going to introduce a new way to solve this game.
Nash Equilibrium

Definition
A Nash Equilibrium is a set of strategies for each player where no change by one player alone can improve his/her outcome; each player has no incentive to change his/her rather stable strategy.

- We are considering Nash Equilibrium where players do not randomize between two or more strategies (called Pure Strategy Nash Equilibrium).
- We only care about alternating strategies in an individual level, not in a group level where everyone collectively change strategies toward a single strategy.
- A Nash Equilibrium is a law that no one would want to break, even in the absence of an effective police force.
- There may exist multiple Nash Equilibria.

Theorem (Nash)
Every game where each player has a finite number of options, has at least one Nash equilibrium.
Iterated Elimination of Strictly Dominated Strategies vs. Nash Equilibrium

- Strictly dominated strategies cannot be a part of a Nash equilibrium.
- After completing the IESDS, if there exists only one strategy for each player remaining, that strategy set is the unique Nash equilibrium.
- Even if there exists no solutions from the IESDS, there may exist Nash Equilibria.
Prisoner’s Dilemma

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- The solution of the IESDS yield that A Confesses and B Confesses.
- Note that this is also a Nash Equilibrium
Stop Light Example

<table>
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<th>Stop</th>
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<tbody>
<tr>
<td>Go</td>
<td>−5, −5</td>
<td>1, 0</td>
</tr>
<tr>
<td>Stop</td>
<td>0, 1</td>
<td>−1, −1</td>
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• This has two Nash Equilibria
  ▶ A = Go and B = Stop (e.g., when A has a green light)
  ▶ A = Stop and B = Go (e.g., when B has a green light)

• No Nash Equilibrium where both players play the same pure strategy.