COMPSCI 240: Reasoning Under Uncertainty

Andrew Lan and Nic Herndon

University of Massachusetts at Amherst

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Lecture 16: Joint PDFs

A Joint PDF of Multiple RVs

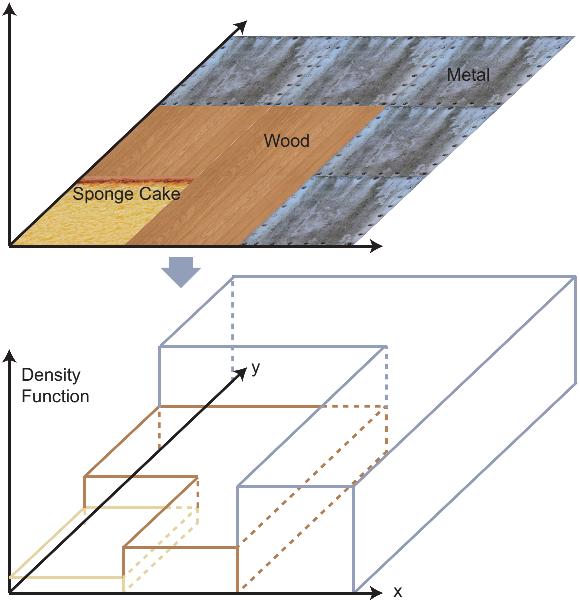
- We now consider a joint PDF of multiple random variables.
- We say that two continuous random variables associated with the same experiment are jointly continuous and have a joint PDF $f_{X,Y}$.

$$P((X,Y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy.$$

- If B is defined such that $B = \{(x,y) | a \leq x \leq b, c \leq y \leq d\}$, then

$$\begin{aligned} P(a \leq x \leq b, c \leq y \leq d) &= \int_c^d \int_a^b f_{X,Y}(x,y)dxdy \\ &= \int_a^b \int_c^d f_{X,Y}(x,y)dydx. \end{aligned}$$

Joint Normal Random Variables



A Joint PDF of Multiple RVs

- A joint PDF should satisfy:
  - **Non-negative**: $f_{X,Y}(x,y) \geq 0$ for all $(X,Y) \subseteq \mathcal{X}^2$
  - **Normalization**: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$.
- We can compute **marginal PDFs** f_X and f_Y as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Example

- Let $f_{X,Y}(x, y)$ be a two-dimensional uniform PDF within $-1 \leq x \leq 1$ and $2 \leq y \leq 6$.

$$f_{X,Y}(x, y) = \begin{cases} c, & \text{if } -1 \leq x \leq 1 \text{ and } 2 \leq y \leq 6 \\ 0, & \text{otherwise,} \end{cases}$$

  Then, what is $P(0 \leq x \leq 1, 2 \leq y \leq 3)$?
- **Solution**: We know that

$$\int_2^6 \int_{-1}^1 c dx dy = 1.$$

- Then, we know that $c = \frac{1}{8}$
- Then,

$$\begin{aligned} P(0 \leq 1 \leq b, 2 \leq y \leq 3) &= \int_2^3 \int_0^1 \frac{1}{8} dx dy \\ &= \frac{1}{8} \end{aligned}$$

Joint CDF

- We define a joint CDF of two RVs X and Y as

$$\begin{aligned} F_{X,Y}(x,y) &= P(X \leq x, Y \leq y) \\ &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)dtds \end{aligned}$$

- Conversely, the joint PDF can be derived from the joint CDF as

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}.$$

Expectation

- If X and Y are random variables, then $Z = g(X, Y)$ is also a random variable.
- The expected value of Z can be computed as

$$E(Z) = E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(X, Y) f_{X,Y}(x, y) dxdy$$

- Note that when $Z = X$, then we can compute the expected value of X.
- If $g(X, Y)$ is a linear function of X and Y, e.g., $g(X, Y) = aX + bY + c$, we have

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

- **Proof:**

Example

Let X and Y are jointly continuous with

$$f(x,y) = \begin{cases} cx^2 + \frac{xy}{3} & \text{if } 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X + Y \geq 1)$.

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(a) Find $P(X + Y \ge 1)$.

$$\int_0^1 \int_0^2 \left(cx^2 + \frac{xy}{3}\right) dydx = 1 \Rightarrow c = 1$$

$$P(X + Y \ge 1) = \int_0^1 \int_{1-x}^2 \left(x^2 + \frac{xy}{3}\right) dydx = \frac{65}{72}$$

(b) Find marginal PDF's of X and Y.

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(b) Find marginal PDF's of X and Y.

$$f_X(x) = 2x^2 + \frac{2x}{3} \text{ if } 0 \le x \le 1.$$

$$f_Y(y) = \frac{1}{3} + \frac{y}{6} \text{ if } 0 \le y \le 2.$$

(c) Are X and Y independent?

Example

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(c) Are X and Y independent? No.

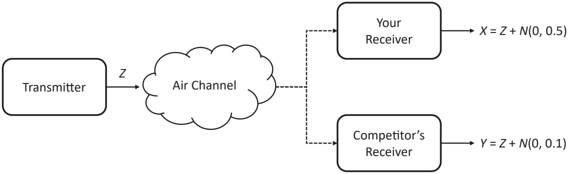
Motivation Example - Covariance and Correlation

- Hypothetically assume that there exists a mysterious wireless signal transmitter that 1) produces a uniform continuous random variable Z from $[0, 5]$ and 2) wirelessly transmits the signal.
- Assume that you are a manufacturer of a new receiver that can estimate the transmitted value of Z with some uncertainty (i.e., noise). Let's say that the noise can be modeled as a normally distributed random variable with mean 0 and standard deviation 0.5. This estimated value X is:

$$X = Z + N(0, 0.5)$$

- Further assume that there exists a competitor in the market that can very accurately estimate the transmitted value of Z. This estimated value Y is:

$$Y = Z + N(0, 0.1)$$



Motivation Example - Covariance and Correlation

- Assume that you, as a new manufacturer, do not know the exact values of these mean and standard deviation, but want to see if your receiver's estimated values agree with the competitor's.
- You collected 1000 values of X and Y through an experiment and compared the values:

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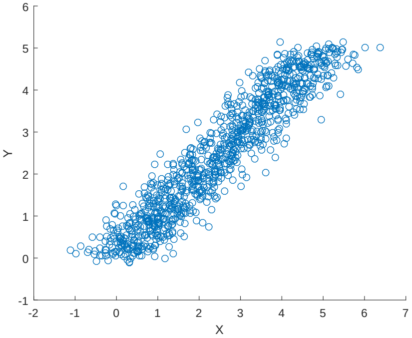

Quantifying Dependence: Covariance

- The **covariance** between any two RVs (either discrete or continuous) X and Y is one measure of dependence that quantifies the degree to which there is a **linear relationship** between X and Y.

$$\begin{aligned}\text{cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

- If X and Y are independent then $cov(X, Y) = 0$.
- However, $cov(X, Y) = 0$ does not necessarily imply that X and Y are independent (see Example 4.13 of the text).
- Note that $\text{cov}(X, X) = \text{var}(X)$.
- For a constant a, $\text{cov}(X, aY + b) = a \cdot \text{cov}(X, Y)$. Prove it.
- Note that $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$. Prove it.

Quantifying Dependence: Covariance

- **Prove** that $\text{cov}(X, Y + Z) = \text{cov}(X, Y) + \text{cov}(X, Z)$.
- More generalized equation

$$\text{cov}\left(X, \sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \text{cov}(X, Y_i)$$

Example

P(X,Y)		
X\Y	$Y=0$	$Y=1$
$X=0$	0.4	0.1
$X=1$	0.2	0.3

- $P(X=0)=0.5, P(X=1)=0.5$ and so $E[X]=0.5$
- $P(Y=0)=0.6, P(Y=1)=0.4$ and so $E[Y]=0.4$
- $E[XY]$ can be computed as follows

$$\begin{aligned} E[XY] &= 0\times 0\times P(X=0,Y=0)+0\times 1\times P(X=0,Y=1) \\ &+ 1\times 0\times P(X=1,Y=0)+1\times 1\times P(X=1,Y=1) \\ &\quad = 0.3 \end{aligned}$$

- $cov(X,Y)=E[XY]-E[X]E[Y]=0.3-0.5\times 0.4=0.1$
- How well X and Y are correlated given that $cov(X,Y)=0.1$?

Quantifying Dependence: Covariance

- Similarly, the computed (empirical) covariance of the previous example was $cov(X, Y) = 2.14$.
- What does this mean?