COMPSCI 240: Reasoning Under Uncertainty

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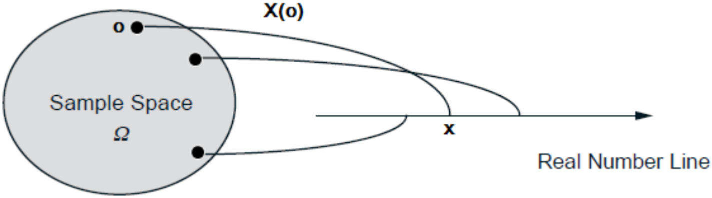
Lecture 12: Multiple Random Variables

Recall: Random Variable

- A *random variable* is a function that maps from the sample space to the real numbers,

$$X : \Omega \to \mathbb{R}$$



Multiple Random Variables

- Consider two random variables, X and Y associated with the same experiment.
- For $x, y \in \mathbb{R}$, we can define events of the form

$$\{X = x, Y = y\} = \{X = x\} \cap \{Y = y\}$$

- The probabilities of these events give the **joint PMF** of X and Y:

$$p_{X,Y}(x, y) = P(X = x, Y = y) = P(X = x \text{ and } Y = y) = P(\{X = x\} \cap \{Y = y\})$$

- Useful for describing **multiple properties** over the outcome space of a single experiment, e.g., pick a random student and let X be their height and Y be their weight.

Tabular Representation of Joint PMFs

P(X,Y)				
X\Y	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$
$X = 1$	0.1	0.1	0	0.2
$X = 2$	0.05	0.05	0.1	0
$X = 3$	0	0.1	0.2	0.1

- e.g., $P(X = 2, Y = 3) = ?$, $P(X = 3, Y = 1) = ?$, ...
- Given the joint PMF, can we compute $P(X = x)$ and $P(Y = y)$?

$$p_X(x) = P(X = x) = \sum_y P(X = x, Y = y)$$

$$p_Y(y) = P(Y = y) = \sum_x P(X = x, Y = y)$$

- If we start with the joint PMF of X and Y, we say $p_X(x)$ is the **marginal PMF** of X and $p_Y(y)$ is the **marginal PMF** of Y.

Computing Marginals from the Joint Distribution

- Suppose Y takes the values $y_1, y_2, \ldots, y_N$, then

$$\{Y = y_1\}, \{Y = y_2\}, \ldots, \{Y = y_N\}$$

  form partitions of Ω_Y.
- Hence, $\{X = x\}$ can be partitioned into

$$\{X = x\} \cap \{Y = y_1\}, \{X = x\} \cap \{Y = y_2\}, \ldots, \{X = x\} \cap \{Y = y_N\}$$

- Therefore,

$$\begin{aligned} P(X = x) &= P(\{X = x\}) \\ &= P(\{X = x\} \cap \{Y = y_1\}) + P(\{X = x\} \cap \{Y = y_2\}) \\ &\qquad \ldots + P(\{X = x\} \cap \{Y = y_N\}) \\ &= \sum_y P(\{X = x\} \cap \{Y = y\}) = \sum_y P(X = x, Y = y) \end{aligned}$$

Marginal PMFs

P(X,Y)				
X\Y	1	2	3	4
1	0.1	0.1	0	0.2
2	0.05	0.05	0.1	0
3	0	0.1	0.2	0.1

X	P(X)
1	0.4
2	0.2
3	0.4

Marginal PMFs

P(X,Y)				
X\Y	1	2	3	4
1	0.1	0.1	0	0.2
2	0.05	0.05	0.1	0
3	0	0.1	0.2	0.1

Y	1	2	3	4
P(Y)	0.15	0.25	0.3	0.3

Example 1

P(X,Y)				
X\Y	1	2	3	4
1	0.1	0.1	0	0
2	0	0.05	0.1	0.05
3	0.1	0.2	0.2	0.1

What's the value of $P(X = 2, Y = 3)$?

A: 0

B: 0.1

C: 0.05

D: 0.2

E: 1

Answer is B.

Example 2

P(X,Y)				
X\Y	1	2	3	4
1	0.1	0.1	0	0
2	0	0.05	0.1	0.05
3	0.1	0.2	0.2	0.1

What's the value of $P(X = 3)$?

A: 0.1

B: 0.4

C: 0.05

D: 0.6

E: 1

Answer is D.

Conditional PMFs

- Conditional PMF of X given Y:

$$P(X = i|Y = j) = P(\{X = i\}|\{Y = j\}) \ .$$

- Compute $P(X|Y)$ using the definition of conditional probability:

$$P(X = i|Y = j) = \frac{P(X = i, Y = j)}{P(Y = j)}$$

  since for any two events A, B we have $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
- The conditional probability $P(X = i|Y = j)$ is the joint probability $P(X = i, Y = j)$ normalized by the marginal $P(Y = j)$.
- An equivalent definition of independence is X and Y are independent if

$$\text{for all } i, j, \quad P(X = i|Y = j) = P(X = i)$$

Conditional PMFs

P(X,Y)				
X\Y	1	2	3	4
1	0.1	0.1	0	0.2
2	0.05	0.05	0.1	0
3	0	0.1	0.2	0.1

Y	1	2	3	4
P(Y)	0.15	0.25	0.3	0.3

$P(X\|Y)$				
X \Y	1	2	3	4
1	0.66	0.4	0	0.66
2	0.33	0.2	0.33	0
3	0	0.4	0.66	0.33

Functions of Two Random Variables

Given two random variables X and Y and a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$Z = f(X, Y)$$

is a new random variable. For example, pick random students and let X be their height and Y be their weight. If we define Z as the Body Mass Index (BMI) where

$$\text{BMI} = \text{weight (lb)}/(\text{height (in)})^2 \times 703.$$

That is,

$$Z = f(X, Y) = Y/X^2 \times 703.$$

Then, Z is also a random variable.

Functions of Two Random Variables

The PMF of Z can be expressed as

$$p_Z(z) = \sum_{\{(x,y)|f(x,y)=z\}} p_{X,Y}(x,y).$$

For example, let us define a new random variable $Z = X \times Y$ where the joint PMF of X and Y is

P(X,Y)				
X\Y	1	2	3	4
1	0.1	0.1	0	0
2	0	0.05	0.1	0.05
3	0.1	0.2	0.2	0.1

Then, the PMF of Z looks like

Z	1	2	3	4	6	8	9	12
$P(Z)$	0.1	0.1	0.1	0.05	0.3	0.05	0.2	0.1

Expectation and Variance of Two Random Variables

- The expected value and variance of Z can be respectively computed as

$$E(Z) = \sum_z zP(Z = z) = \sum_{x,y} f(x, y)P(X = x, Y = y)$$

$$= \sum_x \sum_y f(x, y)P(X = x, Y = y)$$

$$= \sum_y \sum_x f(x, y)P(X = x, Y = y)$$

and

$$var(Z) = E(Z^2) - (E(Z))^2.$$

- If X and Y are **independent**, for all x, y

$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

Linearity of Expectation

- **Lemma:** Given two random variables X, Y, and $Z = X + Y$ then
$$E[Z] = E[X + Y] = E[X] + E[Y]$$

- **Proof:** Generalized expected value rule.

$$\begin{aligned}
E[Z] &= \sum_a \sum_b (a+b) \cdot P(X=a, Y=b) \\
&= \sum_a \sum_b a \cdot P(X=a, Y=b) + \sum_a \sum_b b \cdot P(X=a, Y=b) \\
&= \sum_a a \sum_b P(X=a, Y=b) + \sum_b b \sum_a P(X=a, Y=b) \\
&= \sum_a aP(X=a) + \sum_b bP(Y=b) \\
&= E(X) + E(Y)
\end{aligned}$$

Expectation of Products of Independent Variables

- **Lemma:** If X and Y are independent then $E[XY] = E[X]E[Y]$:
- **Proof:**

$$
\begin{aligned}
E[XY] &= \sum_a \sum_b ab \cdot P(X = a, Y = b) \\
&= \sum_a \sum_b ab \cdot P(X = a)P(Y = b) \\
&= \sum_a a \cdot P(X = a) \cdot \sum_b b \cdot P(Y = b) \\
&= E[X]E[Y]
\end{aligned}
$$

Variance of Sums of Random Variables

- **Lemma:** If X and Y are independent then

$$var(X+Y) = var(X) + var(Y)$$

- **Proof:**

$$\begin{aligned} var(X+Y) &= E[(X+Y)^2] - E[X+Y]^2 \\ &= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\ &= E[X^2] + 2E[X]E[Y] + E[Y^2] \\ &\quad - (E[X]^2 + 2E[X]E[Y] + E[Y]^2) \\ &= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 \\ &= var(X) + var(Y) \end{aligned}$$