COMPSCI 240: Reasoning Under Uncertainty

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Lecture 11: Functions of Random Variables

Recap: Expected Value

- For a random variable X, the expected value is defined to be:

$$E[X] = \sum_{x \in \mathbb{R}} x \, P(X = x)$$

  i.e., the probability-weighted average of the possible values of X.
- $E[X]$ is also called the **expectation** or **mean** of X.
- Why do we care to know about the expected value?
- Given a certain PMF, what is the "average" outcome that I am expecting to have?
- For example, if I bet the same amount of money on roulette and play it for a long-term period, how much do I expect to make?
- For a long-term period, can you make money from casino?

Recap: Variance

- **Definition**: Variance measures how far we expect a random variable to be from its average:

$$\text{var}(X) = E[(X - E[X])^2] = \sum_k (k - E[X])^2 \cdot P(X = k)$$

- An equivalent definition is

$$\text{var}(X) = E[X^2] - E[X]^2$$

- **Definition**: we generally define the $\mathbf{n^{th}}$ **moment** of X as $E[X^n]$, the expected value of the random variable X^n.

Variance of Common Random Variables

- **Bernoulli:** $\text{var}[X] = p(1-p)$
- **Binomial:** $\text{var}[X] = np(1-p)$
- **Geometric:** $\text{var}[X] = \frac{1-p}{p^2}$
- **Uniform:** $\text{var}[X] = \frac{(b-a+1)^2-1}{12}$
- **Poisson:** $\text{var}[X] = \lambda$

Exercise: Variance of the Bernoulli

- PMF of the Bernoulli can be written as

$$p_X(k) = \begin{cases} p, \text{ if } k = 1 \\ 1 - p, \text{ if } k = 0 \end{cases}$$

- $\text{var}(X) = E[X^2] - E[X]^2$ where
- $E[X] = 1 \times p + 0 \times (1 - p) = p$ and $E[X^2] = 1^2 \times p + 0^2 \times (1 - p) = p$.
- Thus, $\text{var}(X) = p - p^2$.

Standard Deviation

- The term **standard deviation** simply refers to the positive square root of the variance, which always exists and is also positive:

$$\text{std}(X) = \sqrt{\text{var}(X)}$$

- The standard deviation is **also** a measure of dispersion around the mean.
- One reason that people like to report standard deviations instead of variances is that **the units are the same as** X.
- $\text{var}(X) = E[(X - E[X])^2]$ vs. $\text{std}(X) = \sqrt{E[(X - E[X])^2]}$
- Example 1: If X is height in feet, then $\text{var}(X)$ has units in square feet while $\text{std}(X)$ again has units in feet.

Functions of Random Variables

- If X is a random variable and $f : \mathbb{R} \to \mathbb{R}$ then

$$Y = f(X)$$

is also a random variable with PMF:

$$P(Y = k) = P(f(X) = k) = \sum_{o \in \Omega \text{ with } f(X(o))=k} P(o)$$

- Example, let X represent an outcome from a 6 sided fair die where

$$P(X = i) = 1/6, \forall i.$$

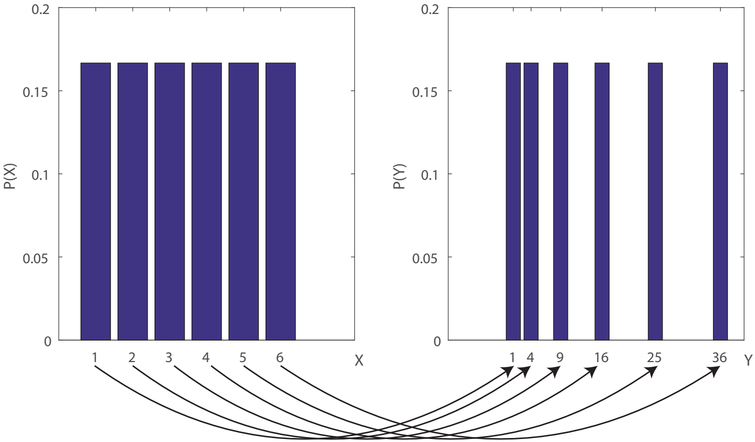
Suppose that you will receive money that is the square of the outcome, and we define a r.v. Y as the amount of money.

- This function $Y = f(X)$ can be expressed as

$$Y = X^2$$

Functions of Random Variables

- Note that Y is also a random variable, whose PMF looks like.



Functions of Random Variables

- Expectation of $Y = f(X)$:

$$E[Y] = \sum_y yP(Y = y) = \sum_y yP(X = f^{-1}(y))$$

$$= \sum_x f(x)P(X = x)$$

- For the previous example,

$$E[X] = 1 \times 1/6 + 2 \times 1/6 + \cdots + 6 \times 1/6 = 3.5$$

$$E[Y] = 1^2 \times 1/6 + 2^2 \times 1/6 + \cdots + 6^2 \times 1/6 = \$15.2$$

Functions of Random Variables

- Variance of Y:

$$var[Y] = E[(Y - E[Y])^2] = \sum_{k \in \{1,4,\ldots,36\}} (k - E[Y])^2 \cdot P(Y = k)$$

$$= \sum_{k \in \{1,4,\ldots,36\}} (k - E[Y])^2 \cdot P(X = f^{-1}(k))$$

$$= (1 - E[Y])^2 P(X = \sqrt{1}) + (4 - E[Y])^2 P(X = \sqrt{4}) + \cdots$$

- For the previous example,

$$E[X] = 1 \times 1/6 + 2 \times 1/6 + \cdots + 6 \times 1/6 = 3.5$$

$$E[Y] = 1^2 \times 1/6 + 2^2 \times 1/6 + \cdots + 6^2 \times 1/6 = \$15.2$$

- Then, the variance for Y is

$$var[Y] = (1^2 - 15.2)^2 \times 1/6 + (2^2 - 15.2)^2 \times 1/6 + \cdots$$

$$+(6^2 - 15.2)^2 \times 1/6 = 149.1$$

Example: Linear function

- If $Y = aX + b$, then $E[Y] = aE[X] + b$ and $\text{var}[Y] = a^2\text{var}[X]$

$$
\begin{aligned}
\text{var}[Y] &= \text{var}[aX + b] \\
&= \sum_k (ak + b - E[aX + b])^2 P(X = k) \\
&= \sum_k (ak + b - aE[X] - b)^2 P(X = k) \\
&= \sum_k (ak - aE[X])^2 P(X = k) \\
&= a^2 \sum_k (k - E[X])^2 P(X = k) \\
&= a^2\text{var}[X].
\end{aligned}
$$

Example 2.4: Average Speed vs Average Time

- Expected values often provide a convenient vehicle to optimize a decision making process.
- We actually do this in our real-world life
  - Should I detour at the next intersection to minimize the travel time? (i.e. what is the expected travel time for 1) going straight or 2) detouring?
- **Problem**: If the weather is good (which happens with probability 0.6), Alice walks the 2 miles to class at a speed of $V = 5$ MPH, and otherwise rides her motorcycle at a speed of $V = 30$ MPH. What is the mean of the time T to get to class?

Example 2.4: Average Speed vs Average Time

- **Problem**: If the weather is good (which happens with probability 0.6), Alice walks the 2 miles to class at a speed of $V = 5$ MPH, and otherwise rides her motorcycle at a speed of $V = 30$ MPH. What ist he mean of the time T to get to class?
- We can derive PMF of V as the following.

$$P_V(v) = \begin{cases} 0.6, v = 5 \\ 0.4, v = 30 \end{cases}$$

- The mean of the speed V can be computed as

$$E(V) = 0.6 \times 5 + 0.4 \times 30 = 15 \text{ MPH}$$

- Since $T = D/V$ where $D = 2$

$$T = f(V) = 2/V$$

- Thus,

$$E(T) = \frac{2}{5} \times 0.6 + \frac{2}{30} \times 0.4 = \frac{4}{15} \text{ Hours}$$