CS 240: Reasoning Under Uncertainty
Final Exam: Dec. 19, 2018

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CLOSED BOOK. CALCULATORS OK. OTHER ELECTRONIC DEVICES NOT ALLOWED. PLEASE BE RIGOROUS AND PRECISE AND SHOW YOUR WORK. IF YOU NEED EXTRA SPACE, USE THE BACK OF A PAGE. IF YOU HAVE QUESTIONS DURING THE EXAM, RAISE YOUR HAND. THIS EXAM CONSISTS OF FIVE PROBLEMS THAT CARRY A TOTAL OF 90+10 POINTS, AS MARKED. DURATION: 120 MINUTES. BEST WISHES!

Standard Random Variables:

- **Uniform:** For $k = a, \ldots, b$:

$$P(X=k) = \frac{1}{b-a+1}; \qquad E[X] = \frac{a+b}{2}; \qquad V[X] = \frac{(b-a+1)^2-1}{12}$$

- **Bernoulli:** For $k = 0$ or 1:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases}; \qquad E[X] = p; \qquad V[X] = p(1-p)$$

- **Binomial:** For $k = 0, \ldots, n$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}; \qquad E[X] = np \qquad V[X] = np(1-p)$$

- **Geometric:** For $k = 1, 2, 3, \ldots$

$$P(X=k) = (1-p)^{k-1} \cdot p \qquad E[X] = \frac{1}{p} \qquad V[X] = \frac{1-p}{p^2}$$

- **Poisson:** For $k = 0, 1, 2, \ldots$

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}; \qquad E[X] = \lambda \qquad V[X] = \lambda$$

CONTINUOUS RANDOM VARIABLES:

- **UNIFORM:**

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{IF } a \leq x \leq b \\ 0, & \text{OTHERWISE}, \end{cases}$$

- **EXPONENTIAL:**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{IF } x \geq 0 \\ 0, & \text{OTHERWISE} \end{cases},$$

OTHER USEFUL EQUATIONS:

- **COVARIANCE**:

$$\text{COV}(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

- **CORRELATION**:

$$\rho(X,Y) = \text{CORR}(X,Y) = \frac{\text{COV}(X,Y)}{\sqrt{\text{VAR}(X)\text{VAR}(Y)}}$$

- **INTEGRATION BY PARTS**:

$$\int u dv = uv - \int v du$$

- **EXPONENTIAL FUNCTION**:

$$\int_{-\infty}^{\infty} e^{ax} = \frac{1}{a} e^{ax}$$

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

- **Markov Bound:** FOR AN NON-NEGATIVE RANDOM VARIABLE X,

$$P(X \geq c) \leq \frac{E(X)}{c}$$

- **Chebyshev Bound:** FOR A RANDOM VARIABLE X,

$$P(|X - E(X)| \geq c) \leq \frac{Var(X)}{c^2}$$

- **THE STANDARD NORMAL TABLE:**

$\Phi(x)$	0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	.5	.50399	.50798	.51197	.51595	.51994	.52392	.5279	.53188	.53586
.1	.53983	.5438	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
.3	.61791	.62172	.62552	.6293	.63307	.63683	.64058	.64431	.64803	.65173
.4	.65542	.6591	.66276	.6664	.67003	.67364	.67724	.68082	.68439	.68793
.5	.69146	.69497	.69847	.70194	.7054	.70884	.71226	.71566	.71904	.7224
.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.7549
.7	.75804	.76115	.76424	.7673	.77035	.77337	.77637	.77935	.7823	.78524
.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.8665	.86864	.87076	.87286	.87493	.87698	.879	.881	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.9032	.9049	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.9222	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.9452	.9463	.94738	.94845	.9495	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.9608	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.9732	.97381	.97441	.975	.97558	.97615	.9767
2	.97725	.97778	.97831	.97882	.97932	.97982	.9803	.98077	.98124	.98169
2.1	.98214	.98257	.983	.98341	.98382	.98422	.98461	.985	.98537	.98574
2.2	.9861	.98645	.98679	.98713	.98745	.98778	.98809	.9884	.9887	.98899
2.3	.98928	.98956	.98983	.9901	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.9918	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.9943	.99446	.99461	.99477	.99492	.99506	.9952
2.6	.99534	.99547	.9956	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.9972	.99728	.99736
2.8	.99744	.99752	.9976	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.999
3.1	.99903	.99906	.9991	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.9994	.99942	.99944	.99946	.99948	.9995
3.3	.99952	.99953	.99955	.99957	.99958	.9996	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.9997	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.9998	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.9999	.9999	.9999	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997
4	.99997	.99997	.99997	.99997	.99997	.99997	.99998	.99998	.99998	0.99998

Problem 1: (16 points)
Part 1.1 ($4 + 4$ points) In a class of 55 students, 40 students are taking maths course and 35 are taking physics course. Also we know that 22 students are taking both maths and physics courses.

1) (4 points) How many students are taking only physics but not math course?
2) (4 points) How many students are not taking either maths or physics course?

Part 1.2 (8 points) 5 women, 3 men and 3 children are standing in a line. How many possible ways are there to align them in a line while no any two men are next to each other.

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Problem 2: (22 points)

Part 2.1 (12 points) A random variable has the following probability mass function (PMF):

$$P(X = k) = \frac{c}{k}, \quad k = 1, 2, 3, 4, 5, 6,$$

where c is a constant such that the PMF is valid.

1) (3 points) What is the value of c?
2) (3 points) What is $E[X]$?
3) (3 points) What is $\mathrm{Var}[X]$?
4) (3 points) Consider the random variable Y with the following PMF:

$$P(Y = k) = \left(e^{-2}\right) \frac{2^k}{k!}, k = 0, 1, \ldots.$$

Is $E[X]$ greater than $E[Y]$?

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Part 2.2 (10 points) Consider two independent random variables A and B, each can take two possible values $\{1, 2\}$. Let us further assume that $P(A = 1) = P(A = 2) = P(B = 1) = P(B = 2) = \frac{1}{2}$. Let $X = A + B$ be the sum of the two values and let $Y = A \times B$ be the product of the two values.

1) (2 points) What is the value of $P(X = i)$ for all i in the sample space.
2) (2 points) What is the value of $P(Y = j)$ for all j in the sample space.
3) (4 points) What is the value of $P(X = i, Y = j)$ for all (i, j) in the joint sample space.
4) (2 points) Are X and Y independent?

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Problem 3: (8 points)
Prove the following equality:

$$\binom{n}{r} = \binom{n-1}{r-1} \times \frac{n}{r}.$$

Problem 4: (10 points)
From past experience, it is known that the number of tickets purchased by a fan standing in line at the ticket window for a baseball match of Boston Red Sox against New York Yankees (Go! Sox!) follows a distribution that has mean $\mu = 2.4$ and standard deviation $\sigma = 2.0$. Suppose that few hours before the start of one of these matches, there are 100 eager fans standing in line to purchase tickets. If only 250 tickets remain, what is the probability that all 100 fans will be able to purchase the tickets they desire?

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Problem 5: (17 points)

Part 5.1 (7 points) A Markov chain has the following transition matrix:

$$\begin{bmatrix} 2/5 & 1/10 & 1/5 & 1/5 & 1/10 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 2/3 & 0 & 1/3 & 0 \end{bmatrix}$$

1) (3 points) Does this Markov chain have a single recurrent class?

2) ($2 + 2 = 4$ points) Does this Markov chain have a steady state and why?

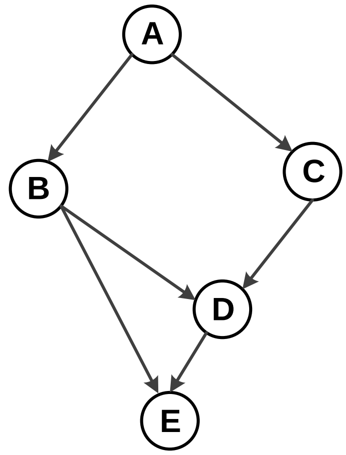
Part 5.2 (10 points) Consider a Markov chain with the following transition matrix:

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{bmatrix}$$

1) (4 points) If the initial distribution of the 3 states S_1, S_2 and S_3 is $v_0 = \langle 1, 0, 0 \rangle$, what is the probability that the chain is in state S_2 after 2 steps.

2) (6 points) This Markov chain has a steady state. Compute the steady state distribution.

Problem 6: (17 points)
You are given with the following Bayesian network consisting of 5 random variables A, B, C, D, E.



Each random variable (A, B, C, D, E) has two possible outcomes: 0 and 1. The probabilities are given below:

$$P(A = 1) = 0.65$$

$$P(B = 1|A) = \begin{cases} 0.2 & \text{when } A = 0 \\ 0.7 & \text{when } A = 1 \end{cases}$$

$$P(C = 1|A) = \begin{cases} 0.6 & \text{when } A = 0 \\ 0.3 & \text{when } A = 1 \end{cases}$$

$$P(D = 1|B, C) = \begin{cases} 0.2 & \text{when } B = 0, C = 0 \\ 0.7 & \text{when } B = 1, C = 0 \\ 0.5 & \text{when } B = 0, C = 1 \\ 0.8 & \text{when } B = 1, C = 1 \end{cases}$$

$$P(E = 1|B, D) = \begin{cases} 0.7 & \text{when } B = 0, D = 0 \\ 0.2 & \text{when } B = 1, D = 0 \\ 0.8 & \text{when } B = 0, D = 1 \\ 0.4 & \text{when } B = 1, D = 1 \end{cases}$$

1) (3 points) Express the joint probability of A, B, C, D, E in a factorized form.

2) (6 points) Compute $P(E = 1|B = 1, C = 1, D = 1)$.

3) (3 points) What is the minimum number of probability numbers we need to store in order to compute the joint probability of A, B, C, D, E?

4) (5 points) If A, B, C, D, E had three outcomes instead of two (e.g., A, B, C, D, E can be either -1, 0, and 1), what is the minimum number of probability numbers we need to store in order to compute the joint probability of A, B, C, D, E?

Extra Credit Problem (10 points)
Suppose that X is a continuous random variable with the following probability density function:

$$f_X(x) = \begin{cases} 6x(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that a value of X divides the range of X (i.e., $[0, 1]$) into two parts: x and $1 - x$. For example, a value of $X = 0.4$ divides the range into 0.4 and 0.6. Similarly, a value of $X = 0.7$ divides the range into 0.7 and 0.3.

Let us now define Y be the ratio of the longer part to the shorter part. For example, when $X = 0.4$, $Y = \frac{0.6}{0.4}$. Similarly, when $X = 0.7$, $Y = \frac{0.7}{0.3}$.

1) What is the probability that Y will be smaller than 2?
(**hint**: you can solve this without computing the probability density function of Y)

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2) Provide a closed-form mathematical expression for the cumulative distribution function of Y: $F_Y(y) = P(Y < y)$.

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	Points Available	Points Achieved
Problem 1 :	16	
Problem 2 :	22	
Problem 3 :	8	
Problem 4 :	10	
Problem 5 :	17	
Problem 6 :	17	
Extra Problem :	10	
Totals :	90 (+10)	