CS 240: Reasoning Under Uncertainty
Exam II: Nov. 14, 2018

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CLOSED BOOK. CALCULATORS OK. OTHER ELECTRONIC DEVICES NOT ALLOWED. PLEASE BE RIGOROUS AND PRECISE AND SHOW YOUR WORK. IF YOU NEED EXTRA SPACE, USE THE BACK OF A PAGE. IF YOU HAVE QUESTIONS DURING THE EXAM, RAISE YOUR HAND. THIS EXAM CONSISTS OF FIVE PROBLEMS THAT CARRY A TOTAL OF 50 POINTS, AS MARKED. DURATION: 60 MINUTES. BEST WISHES!

Standard Random Variables:

- **Uniform:** For $k = a, \ldots, b$:

$$P(X=k) = \frac{1}{b-a+1}; \qquad E[X] = \frac{a+b}{2}; \qquad V[X] = \frac{(b-a+1)^2-1}{12}$$

- **Bernoulli:** For $k = 0$ or 1:

$$P(X=k) = \begin{cases} 1-p & \text{if } k=0 \\ p & \text{if } k=1 \end{cases}; \qquad E[X] = p; \qquad V[X] = p(1-p)$$

- **Binomial:** For $k = 0, \ldots, n$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}; \qquad E[X] = np \qquad V[X] = np(1-p)$$

- **Geometric:** For $k = 1, 2, 3, \ldots$

$$P(X=k) = (1-p)^{k-1} \cdot p \qquad E[X] = \frac{1}{p} \qquad V[X] = \frac{1-p}{p^2}$$

- **Poisson:** For $k = 0, 1, 2, \ldots$

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}; \qquad E[X] = \lambda \qquad V[X] = \lambda$$

Continuous Random Variables:

- **Uniform:**

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise,} \end{cases}$$

- **Exponential:**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases},$$

Other Useful Equations:

- **Integration by parts**:

$$\int udv = uv - \int vdu$$

- **Exponential Function**:

$$\int_{-\infty}^{\infty} e^{ax} = \frac{1}{a} e^{ax}$$

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

- **Markov Bound:** For an non-negative random variable X,

$$P(X \ge c) \le \frac{E(X)}{c}$$

- **Chebyshev Bound:** For a random variable X,

$$P(|X - E(X)| \ge c) \le \frac{Var(X)}{c^2}$$

- **The Standard Normal Table:**

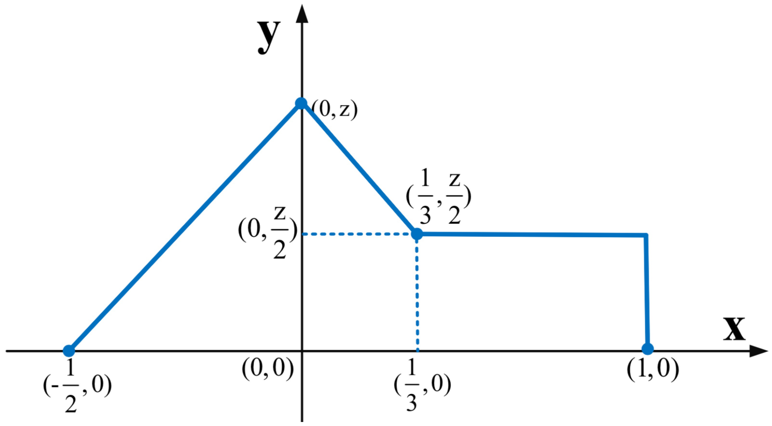
$\Phi(x)$	0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	.5	.50399	.50798	.51197	.51595	.51994	.52392	.5279	.53188	.53586
.1	.53983	.5438	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
.3	.61791	.62172	.62552	.6293	.63307	.63683	.64058	.64431	.64803	.65173
.4	.65542	.6591	.66276	.6664	.67003	.67364	.67724	.68082	.68439	.68793
.5	.69146	.69497	.69847	.70194	.7054	.70884	.71226	.71566	.71904	.7224
.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.7549
.7	.75804	.76115	.76424	.7673	.77035	.77337	.77637	.77935	.7823	.78524
.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.8665	.86864	.87076	.87286	.87493	.87698	.879	.881	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.9032	.9049	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.9222	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.9452	.9463	.94738	.94845	.9495	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.9608	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.9732	.97381	.97441	.975	.97558	.97615	.9767
2	.97725	.97778	.97831	.97882	.97932	.97982	.9803	.98077	.98124	.98169
2.1	.98214	.98257	.983	.98341	.98382	.98422	.98461	.985	.98537	.98574
2.2	.9861	.98645	.98679	.98713	.98745	.98778	.98809	.9884	.9887	.98899
2.3	.98928	.98956	.98983	.9901	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.9918	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.9943	.99446	.99461	.99477	.99492	.99506	.9952
2.6	.99534	.99547	.9956	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.9972	.99728	.99736
2.8	.99744	.99752	.9976	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.999
3.1	.99903	.99906	.9991	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.9994	.99942	.99944	.99946	.99948	.9995
3.3	.99952	.99953	.99955	.99957	.99958	.9996	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.9997	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.9998	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.9999	.9999	.9999	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997
4	.99997	.99997	.99997	.99997	.99997	.99997	.99998	.99998	.99998	0.99998

Problem 1: $(5+5$ points)

1) Let us assume that X is a random variable with the following probability density function. Find the value of a.

$$f_X(x) = \begin{cases} -ax, & -2 \leq x \leq 0 \\ 2ax^2, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

2) Let us assume that X is a random variable whose probability density function is depicted in the following figure. Find the value of z.



Problem 2: (10 points)
Suppose that we have two independent exponential random variables X and Y. The probability density functions of X and Y are:

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & \text{if } 0 \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \beta e^{-\beta y} & \text{if } 0 \leq y \leq \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that X is greater than Y.

Problem 3: $(5 + 5 = 10$ points)

1) Prove the following statement:

$$\mathrm{Var}(X + Y + Z) = \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Var}(Z) + 2\mathrm{Cov}(X, Y) + 2\mathrm{Cov}(X, Z) + 2\mathrm{Cov}(Y, Z)$$

2) Prove the following statement:

$$\operatorname{cov}(X+Y, Z+K) = \operatorname{cov}(X, Z) + \operatorname{cov}(X, K) + \operatorname{cov}(Y, Z) + \operatorname{cov}(Y, K)$$

Problem 5: $(3 + 3 + 6 = 12$ points)

1) Suppose it is known that the number of items produced in a factory during a week is a random variable with mean of 100 and variance of 20. Compute two **upper bounds** for the probability that this week's production will exceed 130 using the Markov's inequality and the Chebyshev's inequality.
(**Hint**: the Chebyshev's bound can also be used as the upper bound. Please show your work to justify that the Chebyshev's bound can be used as the upper bound.)

2) Suppose again that it is known that the number of items produced in a factory during a week is a random variable with mean of 100 and variance of 20. Considering the probability that this week's production will exceed an arbitrary number N, what value range of N would make Markov's bound tighter than the Chebyshev's bound?

3) The mean values of three normal random variables X, Y, Z are 1, 2, and 3, respectively. If $P(3 < X + 3Y - Z < 5) = 0.4$, find $P(0.2X + 0.6Y - 0.2Z < 0.6)$.

	Points Available	Points Achieved
Problem 1 :	10	
Problem 2 :	10	
Problem 3 :	10	
Problem 4 :	8	
Problem 5 :	12	
Totals :	50	

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