Discussion 01

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TAs and UCAs

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Details about HW1 and Quiz1

Discussions occur in this room: LGRC A104A

Quiz 1 available, due Fri, Feb 1

Homework 1 available, due Fri, Feb 8

Helpful Quiz - Set Theory and Probability Theory

Consider a probability law and let A, B be events.

1. $P(A \cup B) = ?$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. $P(A \cap B) = ?$

$$P(A \cap B) = P(B) \cdot P(A|B)$$
$$P(A \cap B) = P(A) \cdot P(B|A)$$

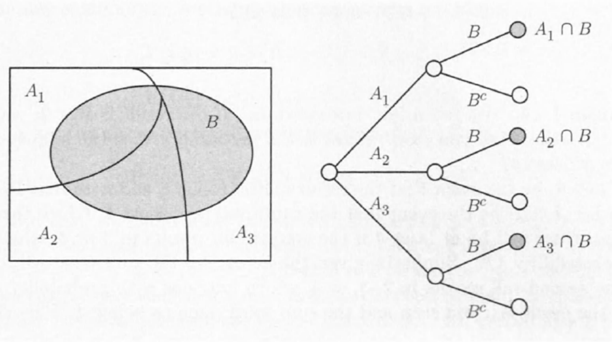
3. $P(A|B) = ?$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Helpful Quiz - Total Probability Theorem

4. **Total Probability Theorem**: Let $A_1, A_2, \ldots, A_n$ form a partition of Ω and $P(A_i) > 0 \;\forall\; i$. Then for any event B, we have $P(B) = ?$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$
$$P(B) = P(A_1)P(B|A_1) + \ldots + P(A_n)P(B|A_n)$$



Helpful Quiz - Bayes' Theorem

5. **Bayes' Theorem**: Let $A_1, A_2, \ldots, A_n$ form a partition of Ω and $P(A_i) > 0 \ \forall \ i$. Then for any event B, such that $P(B) > 0$, we have $P(A_i|B) = ?$

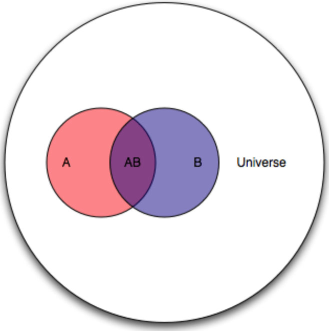
$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}$$

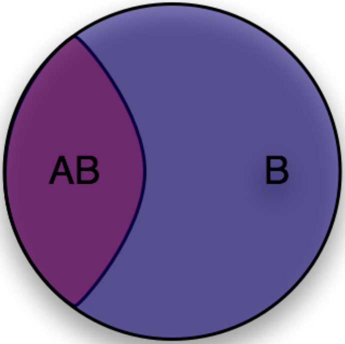
$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + \ldots + P(A_n)P(B|A_n)}$$

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(A_1 \cap B) + \ldots + P(A_n \cap B)}$$

Helpful Quiz - Bayes' Theorem Visualization[1]

$P(A|B) = ?$





[1] https://oscarbonilla.com/2009/05/visualizing-bayes-theorem/

Problem Set

Problem Set - Rules

1. Must be completed in groups of 3-5.
2. Problems should only take a couple of minutes.
3. Discussion between group members is encouraged.
4. Feel free to ask me questions as well!
5. Note: $A \setminus B =$ the *set difference* of A minus B, all elements of A that are not also elements of B.

Problem Set - Problem #1

Recall

Suppose Ω is the sample space and P is the probability rule. Then, for any event $A = \{e_1, e_2, \ldots, e_t\}$ we have:
$P(A) = P(\{e_1\}) + P(\{e_2\}) + \ldots + P(\{e_t\})$.

Note that because $P(\Omega) = 1$, if all atomic events have the same probability and there are r atomic events, each atomic event must have probability $\frac{1}{r}$.

$A \setminus B$ is the set of all elements of A that are not in B.

True or False: For any two events A and B,

$$P(A \cap B) \leq P(A \cup B)$$

Problem 1 – Solution (TRUE)

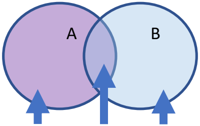
$$\boldsymbol{For\ any\ two\ events\ A\ and\ B, is\ P(A \cup B) \geq P\ (A\ \cap\ B\ )true?}$$

Problem 1 – Solution (TRUE)

$\boldsymbol{For\ any\ two\ events\ A\ and\ B, is\ P(A \cup B) \geq P\ (A\ \cap\ B\ ) true?}$

$A\ \cup\ B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$

The sets are disjoint, so:

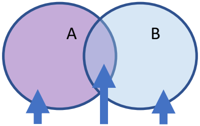


$(A \backslash B) \cup (A \cap B) \cup (B \backslash A)$

Problem 1 – Solution (TRUE)

$$\boldsymbol{For\ any\ two\ events\ A\ and\ B, is\ P(A \cup B) \geq P\ (A\ \cap\ B\ ) true?}$$

$A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$



$(A \backslash B) \cup (A \cap B) \cup (B \backslash A)$

The sets are disjoint, so:

$P(A \cup B) = P(A \backslash B) + P(A \cap B) + P(B \backslash A) \Leftrightarrow P(A \cup B) \geq P(A \cap B)$

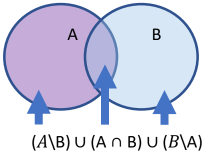
Because the axioms of probability state

Problem 1 – Solution (TRUE)

For any two events A and B, is $P(A \cup B) \geq P(A \cap B)$ ***true?***

$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$

The sets are disjoint, so:



$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A) \iff P(A \cup B) \geq P(A \cap B)$

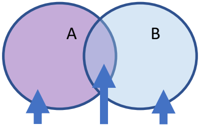
Because the axioms of probability state

$P(A \setminus B) \geq 0$ ***and*** $P(B \setminus A) \geq 0.$ **(Non-negativity axiom (P.15, textbook))**

Problem 1 – Solution (TRUE)

$$\boldsymbol{For\ any\ two\ events\ A\ and\ B, is\ P(A \cup B) \geq P\ (A\ \cap\ B\ ) true?}$$

$$A\ \cup\ B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$$



The sets are disjoint, so:

$$P(A \cup B) = P(A \backslash B) + P(A \cap B) + P(B \backslash A) \iff P(A\ \cup\ B) \geq P(A \cap B)$$

Because the axioms of probability state

$P(A \backslash B) \geq 0\ \boldsymbol{and}\ P(B \backslash A) \geq 0.$ **(Non-negativity axiom (P.15, textbook))**

Alternatively, recall that:

$$\boxed{P(A \cup B) \leq P(A) + P(B)\ \boldsymbol{and}\ P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Problem Set - Problem #2

Recall

Suppose Ω is the sample space and P is the probability rule. Then, for any event $A = \{e_1, e_2, \ldots, e_t\}$ we have: $P(A) = P(\{e_1\}) + P(\{e_2\}) + \ldots + P(\{e_t\})$.

Note that because $P(\Omega) = 1$, if all atomic events have the same probability and there are r atomic events, each atomic event must have probability $\frac{1}{r}$.

$A \setminus B$ is the set of all elements of A that are not in B.

True or False: For any three events A, B, and C,

$$P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$$

Problem 2 – Solution (TRUE)

$\boldsymbol{For\ any\ three\ events\ A, B, and\ C, is\ P(A \cup B \cup C) \leq P(A) + \ P(B) + P(C)\ true?}$

Problem 2 – Solution (TRUE)

$$\boldsymbol{For\ any\ three\ events\ A, B, and\ C, is\ P(A \cup B \cup C) \leq P(A) +\ P(B) + P(C)\ true?}$$

Can use this property on P.14, textbook: $P(A \cup B) \leq P(A) +\ P(B)$

Problem 2 – Solution (TRUE)

$$\boldsymbol{For\ any\ three\ events\ A, B, and\ C, is\ P(A \cup B \cup C) \leq P(A) + P(B) + P(C)\ true?}$$

Can use this property on P.14, textbook: $P(A \cup B) \leq P(A) + P(B)$

To apply to sets $A_1 \cup A_2 \cup \ldots \cup A_n \Longrightarrow P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2 \cup \ldots \cup A_n)$

Problem 2 – Solution (TRUE)

$\boldsymbol{For\ any\ three\ events\ A, B, and\ C, is\ P(A \cup B \cup C) \leq P(A) + P(B) + P(C)\ true?}$

Can use this property on P.14, textbook: $P(A \cup B) \leq P(A) + P(B)$

To apply to sets $A_1 \cup A_2 \cup \ \dots \cup A_n \Longrightarrow P(A_1 \cup A_2 \cup \ \dots \cup A_n) \leq P(A_1) + P(A_2 \cup \ \dots \cup A_n)$

$P(A_1 \cup A_2 \cup \ \dots \cup A_n) \leq P(A_1) + P(A_2 \cup \ \dots \cup A_n) \Longrightarrow P(A_2 \cup \ \dots \cup A_n) \leq P(A_2) + P(A_3 \cup \ \dots \cup A_n)$

Apply again recursively here to get:

$$\boxed{P(A_1 \cup A_2 \cup \ \dots \cup A_n) \leq \sum_{i=1}^{n} P(A_i)}$$

Problem 2 – Solution (TRUE)

$$\boldsymbol{For\ any\ three\ events\ A, B, and\ C, is\ P(A \cup B \cup C) \le P(A) + P(B) + P(C)\ true?}$$

Can use this property on P.14, textbook: $P(A \cup B) \le P(A) + P(B)$

To apply to sets $A_1 \cup A_2 \cup \ldots \cup A_n \Rightarrow P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2 \cup \ldots \cup A_n)$

$P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2 \cup \ldots \cup A_n) \Rightarrow P(A_2 \cup \ldots \cup A_n) \le P(A_2) + P(A_3 \cup \ldots \cup A_n)$

Apply again recursively here to get:

$$\boxed{P(A_1 \cup A_2 \cup \ldots \cup A_n) \le \sum_{i=1}^{n} P(A_i)}$$

Alternatively, we could show that:

$$P(A \cup B \cup C) = P(A \cup (B \cup C))$$
$$P(A \cup B \cup C) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$$
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cup C) - P(A \cap (B \cup C))$$

And we could apply twice from P.15 that:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Problem 2 – Solution (TRUE)

$\boldsymbol{For\ any\ three\ events\ A, B, and\ C, is\ P(A \cup B \cup C) \leq P(A) + P(B) + P(C)\ true?}$

Can use this property on P.14, textbook: $P(A \cup B) \leq P(A) + P(B)$

To apply to sets $A_1 \cup A_2 \cup \ldots \cup A_n \longrightarrow P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2 \cup \ldots \cup A_n)$

$P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2 \cup \ldots \cup A_n) \longrightarrow P(A_2 \cup \ldots \cup A_n) \leq P(A_2) + P(A_3 \cup \ldots \cup A_n)$

Apply again recursively here to get:

$$\boxed{P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \sum_{i=1}^{n} P(A_i)}$$

Alternatively, we could show that:

$$P(A \cup B \cup C) = P(A \cup (B \cup C))$$

$$P(A \cup B \cup C) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cup C) - P(A \cap (B \cup C))$$

And we could apply twice from P.15 that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$-P(B \cup C) - P\big(A \cap (B \cup C)\big) \leq 0\ \boldsymbol{and} + P(B \cup C) + P\big(A \cap (B \cup C)\big) \geq 0$$

(Non-negativity axiom (P.15, textbook))

Such that: $\boxed{P(A \cup B \cup C) \leq P(A) + P(B) + P(C)}$

Problem Set - Problem #3

Recall

Suppose Ω is the sample space and P is the probability rule. Then, for any event $A = \{e_1, e_2, \ldots, e_t\}$ we have: $P(A) = P(\{e_1\}) + P(\{e_2\}) + \ldots + P(\{e_t\})$.

Note that because $P(\Omega) = 1$, if all atomic events have the same probability and there are r atomic events, each atomic event must have probability $\frac{1}{r}$.

$A \setminus B$ is the set of all elements of A that are not in B.

True or False: The probability of getting two sixes when you throw two six-sided dice is larger than the probability of getting five tails in a row when you toss a fair coin five times. *You should assume that all atomic events in the first experiment have the same probability and all atomic events in the second experiment have the same probability*.

Problem 3 – Solution (FALSE)

$$\boldsymbol{P(two\ 6's) > P(5\ Tails)}?$$

Problem 3 – Solution (FALSE)

$$\boldsymbol{P(two\ 6's) > P(5\ Tails)}?$$

$Die:\ A = \{6\}, \qquad \Omega = \{1,2,3,4,5,6\}$ $\qquad$ $Coin: B = \{T\}, \qquad \Omega = \{H, T\}$

Problem 3 – Solution (FALSE)

$$\boldsymbol{P(two\ 6's) > P(5\ Tails)?}$$

$Die\colon\ A = \{6\}, \qquad \Omega = \{1,2,3,4,5,6\}$

$P(A) = \mathrm{P(Getting\ 2\ six's)}$

$= \mathrm{P(Getting\ 1\ six, getting\ 1\ six)}$

$= \mathrm{P(1\ six) * P(1\ six)}$

$$= \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

Problem 3 – Solution (FALSE)

$$\boldsymbol{P(two\ 6's) > P(5\ Tails)}?$$

$Die:\ A = \{6\}, \qquad \Omega = \{1,2,3,4,5,6\}$

$P(A)$ =P(Getting 2 six's)

= P(Getting 1 six, getting 1 six)

= P(1 six) * P(1 six)

$$= \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

$Coin: B = \{T\}, \qquad \Omega = \{H, T\}$

$P(B)$ =P(Getting 5 tails)

=P(T, T, T, T, T)

=P(T)* P(T)* P(T)* P(T)* P(T)

$$= \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2}$$

$$= \frac{1}{2^5} = \frac{1}{32}$$

Problem 3 – Solution (FALSE)

$$\boldsymbol{P(two\ 6's) > P(5\ Tails)}?$$

$Die\colon\ A = \{6\}, \qquad \Omega = \{1,2,3,4,5,6\}$

$P(A) =$P(Getting 2 six's)

$=$ P(Getting 1 six, getting 1 six)

$=$ P(1 six) * P(1 six)

$$= \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

$Coin\colon B = \{T\}, \qquad \Omega = \{H, T\}$

$P(B) =$P(Getting 5 tails)

=P(T, T, T, T, T)

=P(T)* P(T)* P(T)* P(T)* P(T)

$$= \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2}$$

$$= \frac{1}{2^5} = \frac{1}{32}$$

$$\boxed{\boldsymbol{P(two\ 6's) < P(5\ Tails)}}$$

Problem Set - Problem #4

Recall

Suppose Ω is the sample space and P is the probability rule. Then, for any event $A = \{e_1, e_2, \ldots, e_t\}$ we have:
$P(A) = P(\{e_1\}) + P(\{e_2\}) + \ldots + P(\{e_t\})$.

Note that because $P(\Omega) = 1$, if all atomic events have the same probability and there are r atomic events, each atomic event must have probability $\frac{1}{r}$.

$A \setminus B$ is the set of all elements of A that are not in B.

True or False: Consider the experiment where you throw two six-sided dice and observe the two values on the dice. The probability that the sum of the values is odd is equal to the probability that the product of these values is even. *You should assume that all atomic events in the experiment have the same probability*.

Problem 4 – Solution (FALSE)

$$\boldsymbol{P(Sum\ of\ values\ is\ odd) = P(Product\ of\ values\ is\ even)}?$$

Problem 4 – Solution (FALSE)

$$\boldsymbol{P(Sum\ of\ values\ is\ odd) = P(Product\ of\ values\ is\ even)}?$$

$$P(Sum\ of\ values\ is\ odd) = P(1st\ die\ odd, 2nd\ die\ even) + P(1st\ die\ even, 2nd\ die\ odd)$$
$$= P(1st\ O) * P(2nd\ E) + P(1st\ E) * P(2nd\ O)$$
$$= \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = \mathbf{1/2}$$

Problem 4 – Solution (FALSE)

$$\boldsymbol{P(Sum\ of\ values\ is\ odd) = P(Product\ of\ values\ is\ even)}?$$

$$P(Sum\ of\ values\ is\ odd) = P(1st\ die\ odd, 2nd\ die\ even) + P(1st\ die\ even, 2nd\ die\ odd)$$
$$= P(1st\ O) * P(2nd\ E) + P(1st\ E) * P(2nd\ O)$$
$$= \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = \mathbf{1/2}$$

$$P(Product\ of\ values\ is\ even) = P(1st\ O, 2nd\ E) + P(1st\ E, 2nd\ O) + P(1st\ E, 2nd\ E)$$
$$= P(1st\ O) * P(2nd\ E) + P(1st\ E) * P(2nd\ O) + P(1st\ E) * P(2nd\ E)$$
$$= \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = \mathbf{3/4}$$

Problem 4 – Solution (FALSE)

$$\boldsymbol{P(Sum\ of\ values\ is\ odd) = P(Product\ of\ values\ is\ even)}?$$

$$P(Sum\ of\ values\ is\ odd) = P(1st\ die\ odd, 2nd\ die\ even) + P(1st\ die\ even, 2nd\ die\ odd)$$
$$= P(1st\ O) * P(2nd\ E) + P(1st\ E) * P(2nd\ O)$$
$$= \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = \mathbf{1/2}$$

$$P(Product\ of\ values\ is\ even) = P(1st\ O, 2nd\ E) + P(1st\ E, 2nd\ O) + P(1st\ E, 2nd\ E)$$
$$= P(1st\ O) * P(2nd\ E) + P(1st\ E) * P(2nd\ O) + P(1st\ E) * P(2nd\ E)$$
$$= \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = \mathbf{3/4}$$

½ ≠ ¾

$$\boxed{\boldsymbol{P(Sum\ of\ values\ is\ odd) \neq P(Product\ of\ values\ is\ even)}}$$