Discussion 6

3.1 Continuous Random Variables and PDFs

3.2 Cumulative Distribution Functions

3.3 Normal Random Variables

3.4 Joint PDFs of Multiple Random Variables

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March 7, 2019

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Preliminaries

Reminders

1. Moodle quiz #5 is available, due Fri, Mar. 8

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2. HW #2 is graded. You can submit regrade requests within a week.

Quiz 4 Review

Problem #1

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Repeating the toss c times generates a binomial RV, whose expectation is $np = \frac{5}{9}c$

Problem #5

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Two coins are simultaneously tossed until at least one of them comes up a head. The first coin comes up a head with probability p_1, and the second with probability 0.4. All tosses are assumed independent. What is the variance of the number of tosses?

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Variance for geometric RV is $\frac{1-p}{p^2}$.

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A contractor purchases a shipment of 102 transistors. It is his policy to randomly select and test 10 of these transistors and to keep the shipment only if at least 9 of the 10 are in working condition. If we know that 20% of the transistors have defects, what is the probability the contractor will keep all the transistors?

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$$P(X = 9 \text{ or } X = 10) = \binom{10}{9} 0.2^1 0.8^9 + \binom{10}{10} 0.2^0 0.8^{10} = 0.3758$$

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Then $ab = 3.5(2.8 - 0.4 \cdot 3.5) = 4.9$

Practice Problems

Question #1

Problem Statement

Alvin throws darts at a circular target of radius r and is equally likely to hit any point in the target. Let X be the distance of Alvin's hit from the center.

Find the PDF, the mean, and the variance of X.

Hint: Calculate CDF first.

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So

$$Var(X) = E[X^2] - E[X]^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}.$$

Question #2

Problem Statement

Jane goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameters λ.

What is the CDF of Jane's waiting time T?

Hint: use total probability theorem.

$F_T(t) = P(T \leq t) = \cdots$

An exponentially distributed random variable X with parameter λ ($\lambda > 0$) has a PDF of the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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We obtain

$$F_T(t) = \begin{cases} \frac{1}{2}(2 - e^{-\lambda t}) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Helpful Quiz (Time Permitting)

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FIN