Administrivia

- Mini-project 2 (really) due today
  - Turn in a printout of your work at the end of the class
- Project presentations
  - **April 23** (Thursday next week) and **28** (Tuesday the week after)
  - Order will be randomized so be prepared to present on either day
  - **8 mins** total for each group (set aside 1-2 mins for questions)
    - Problem statement, approach, preliminary results, future work
    - How to give a good talk: http://www.cs.berkeley.edu/~jrs/speaking.html
- Final report
  - Maximum of 8 pages in NIPS paper format (word + latex style files)
    - https://nips.cc/Conferences/2014/PaperInformation/StyleFiles
  - Writeup due on **May ~~3~~ 6** (submit pdf via Moodle)
    - Hard deadline — I've to submit your grades to the University

Overview of ML so far …

- Supervised learning
  - decision trees
  - k nearest neighbor
  - perceptrons (+ kernels)
  - neural networks (+ convolution)
- Learning is:
  - optimization
  - density estimation

  … with known labels
- Learning is hard
  - bias-variance tradeoff
  - ensembles reduce variance
- Learning is possible
  - boosting weak learners

- Unsupervised learning
  - k-means
  - PCA (+ kernels)
  - spectral methods
  - mean shift
- Learning is:
  - optimization
  - density estimation

  … with hidden "labels"
- **EM:** a general technique to solve hidden variable problem

- Reinforcement learning
  - labels come from experience
  - guest lecture by Kevin Spiteri next Tuesday

Hidden Markov Models

Subhransu Maji

CMPSCI 689: Machine Learning

16 April 2015

Reasoning over time

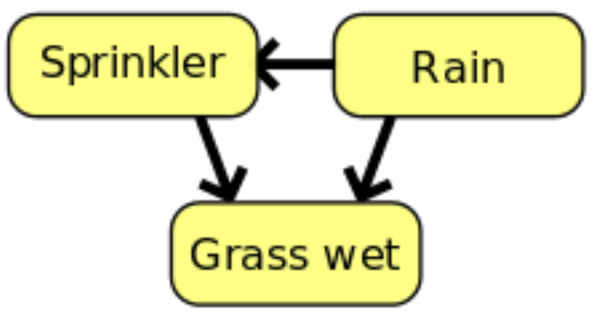
- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
  - Weather forecasting

- Need to introduce time in our models
- Basic approach: Hidden Markov models (HMMs)
- More general approach: Dynamic Bayes Nets (DBNs)

Bayes network — a quick intro

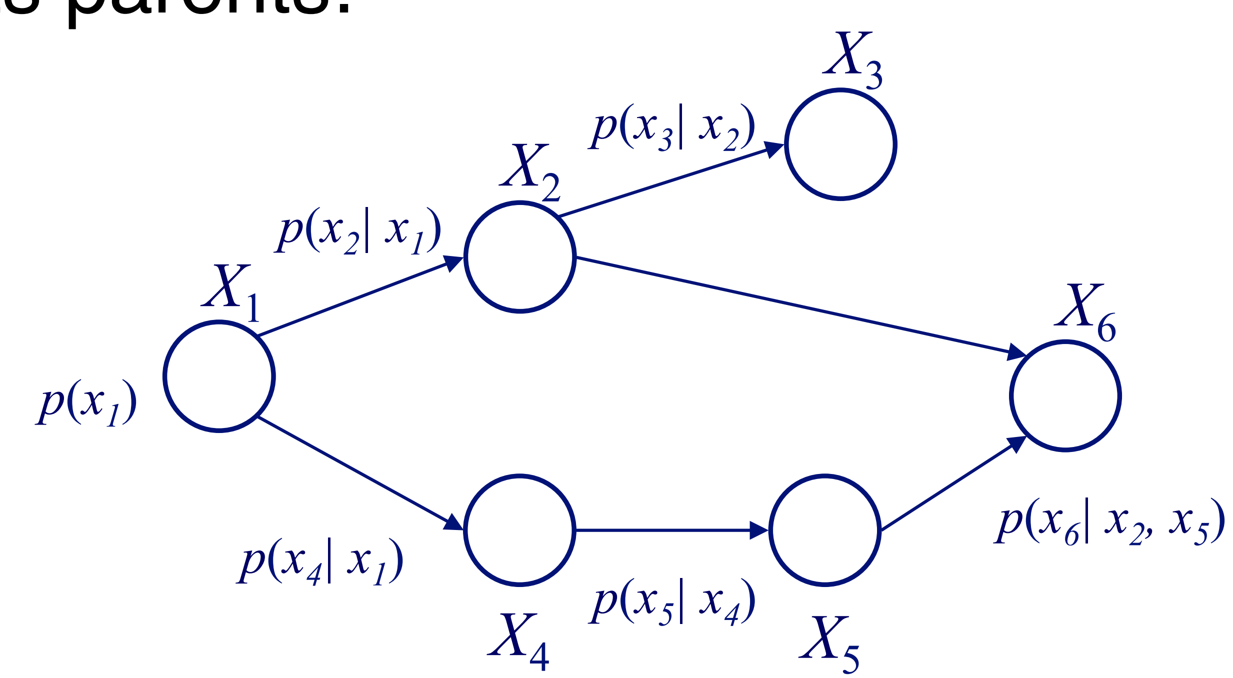
- A way of specifying conditional independences
- A Bayes Network (BN) is a directed acyclic graph (DAG)
- Nodes are random variables
- A node's distribution only depends on its parents
- Joint distribution decomposes:

$$p(x) = \Pi_i p(x_i | \text{Parents}_i)$$



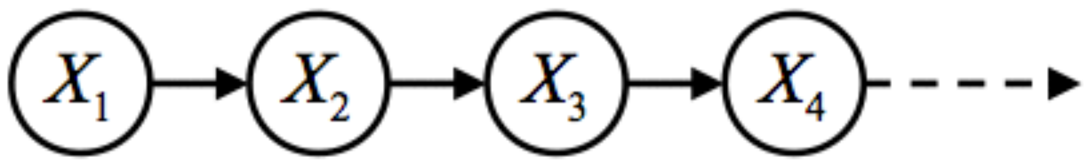
example BN

- A node's value is conditionally independent of everything else given the value of its parents:



Markov models

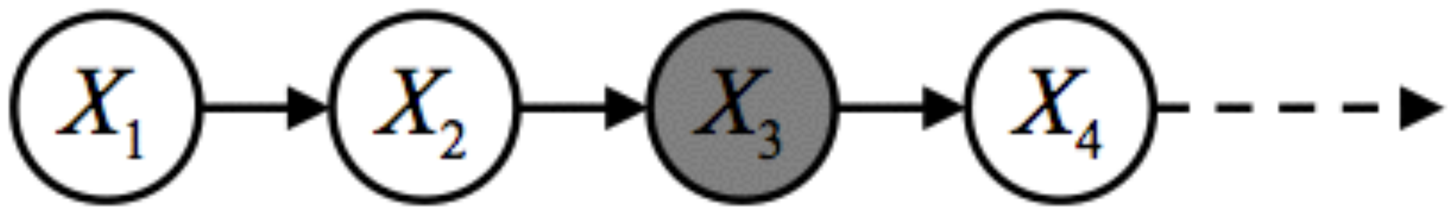
- A Markov model is a chain-structured BN
  - Each node is identically distributed (stationarity)
  - Value of X at a given time t is called the state
  - As a BN:



$$P(X_1) \qquad P(X_2|X_1) \qquad P(X_T|X_{T-1})$$

- Parameters of the model
  - Transition probabilities or dynamics that specify how the state evolves over time
  - The initial probabilities of each state

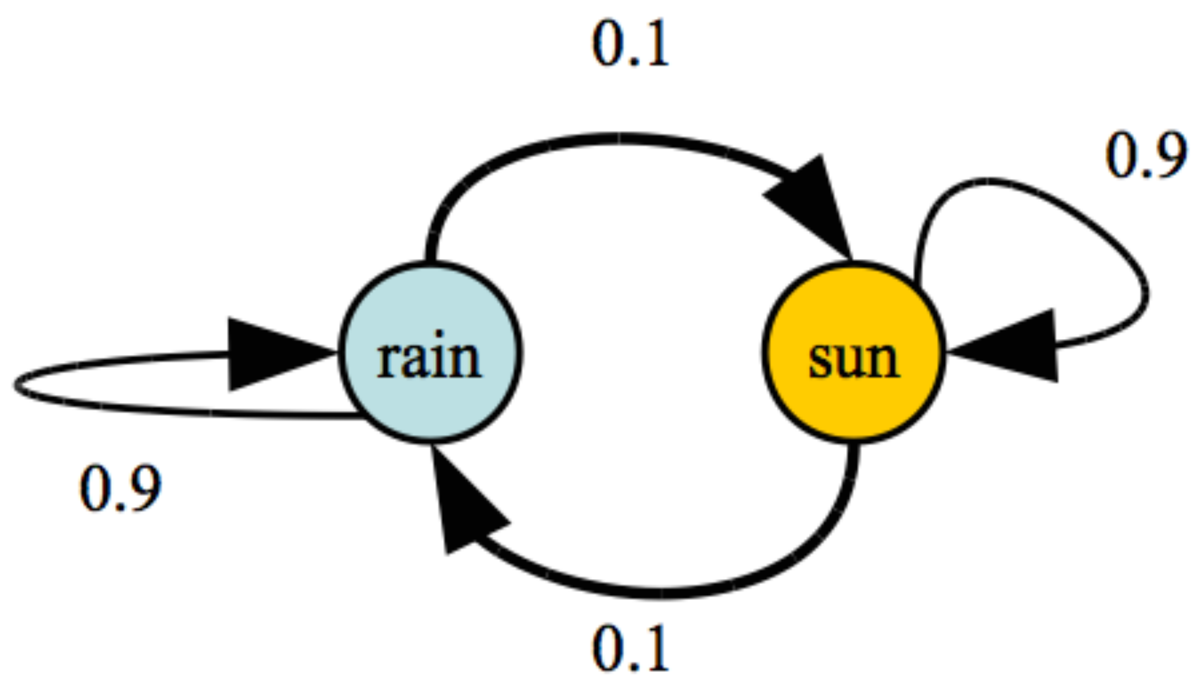
Conditional independence



- Basic conditional independence:
  - Past and future independent of the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property
- Note that the chain is just a (growing) BN
  - We can always use generic BN reasoning on it (if we truncate the chain)

Markov model: example

- Weather:
  - States: X = {rain, sun}
  - Transitions:



This is a CPT and not a BN!

  - Initial distribution: 1.0 sun
  - **Question:** What is the probability distribution after one step?

$$P(X_2=\text{sun}) = P(X_2=\text{sun}|X_1=\text{sun})P(X_1=\text{sun}) + P(X_2=\text{sun}|X_1=\text{rain})P(X_1=\text{rain})$$
$$= 0.9 \times 1.0 + 0.1 \times 0.0$$
$$= 0.9$$

Markov model: example

- Text synthesis — create plausible looking poetry, love letters, term papers, etc.

- Typically a higher order Markov model
- Sample word w_t based on the previous n words i.e:
  - $w_t \sim P(w_t \mid w_{t-1}, w_{t-2}, \ldots, w_{t-n})$
  - These probability tables can be computed from lots of text

- Examples of text synthesis [A.K. Dewdney, Scientific American 1989]
  - “As I've commented before, really relating to someone involves standing next to impossible.”
  - “One morning I shot an elephant in my arms and kissed him.”
  - “I spent an interesting evening recently with a grain of salt”

Slide from Alyosha Efros, ICCV 1999

Mini-forward algorithm

- Question: probability of being in a state x at a time t?
- Slow answer:
  - Enumerate all sequences of length t with end in s
  - Add up their probabilities:

$$P(X_t = \text{sun}) = \sum_{x_1,\ldots,x_{t-1}} P(x_1, \ldots, x_{t-1}, \text{sun})$$

$$P(X_1 = \text{sun})P(X_2 = \text{sun}|X_1 = \text{sun}) \ldots P(X_t = \text{sun}|X_{t-1} = \text{sun})$$

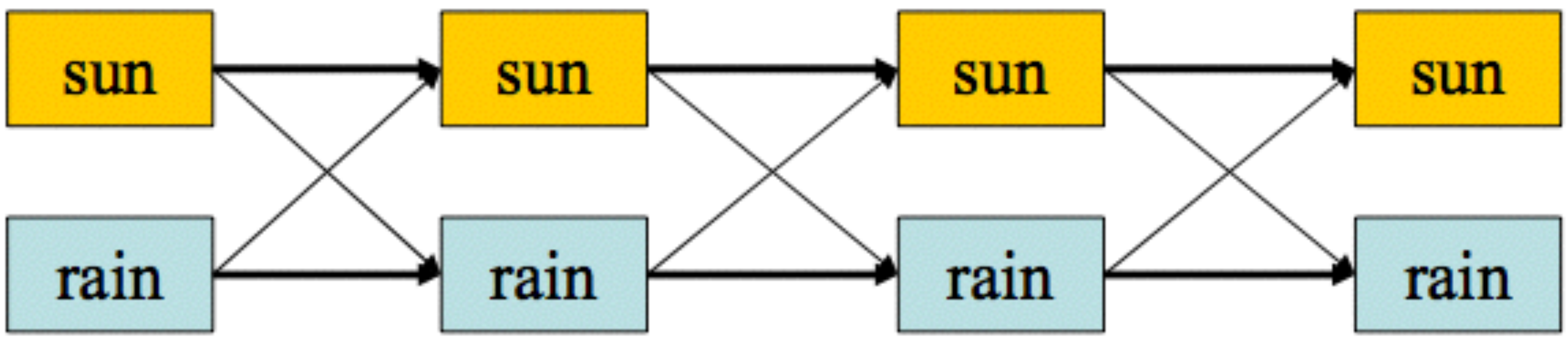
$$P(X_1 = \text{sun})P(X_2 = \text{rain}|X_1 = \text{sun}) \ldots P(X_t = \text{sun}|X_{t-1} = \text{sun})$$

$$\vdots$$

$$O(2^{t-1})$$

Mini-forward algorithm

- Better way: cached incremental belief updates
  - (GM folks: this is an instance of variable elimination)



$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}) P(x_t | x_{t-1})$$

forward simulation

Example

- From initial observation of sun

$$\underset{P(X_1)}{\left\langle \begin{matrix}1.0\\0.0\end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix}0.9\\0.1\end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix}0.82\\0.18\end{matrix} \right\rangle} \quad \Longrightarrow \quad \underset{P(X_\infty)}{\left\langle \begin{matrix}0.5\\0.5\end{matrix} \right\rangle}$$

- From initial observation of rain

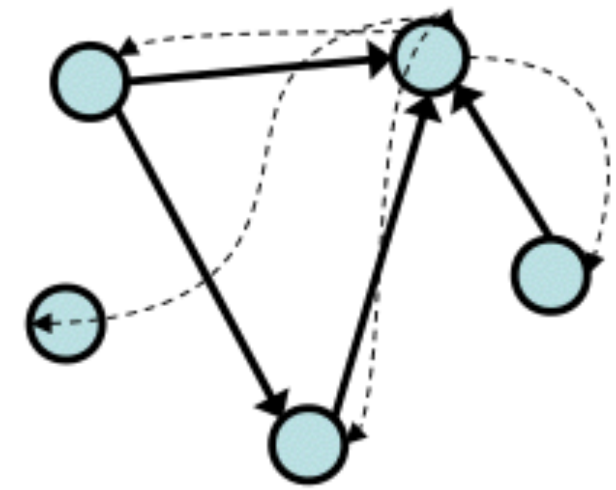
$$\underset{P(X_1)}{\left\langle \begin{matrix}0.0\\1.0\end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix}0.1\\0.9\end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix}0.18\\0.82\end{matrix} \right\rangle} \quad \Longrightarrow \quad \underset{P(X_\infty)}{\left\langle \begin{matrix}0.5\\0.5\end{matrix} \right\rangle}$$

Stationary distribution

- If we simulate the chain long enough
  - What happens?
  - Uncertainty accumulates
  - Eventually, we have no idea what the state is!

- Stationary distributions:
  - For most chains, the distribution we end up in is independent of the initial distribution (but not always uniform!)
  - This distribution is called the stationary distribution of the chain
  - Usually, can only predict a short time out

Web link analysis



PageRank over a web graph

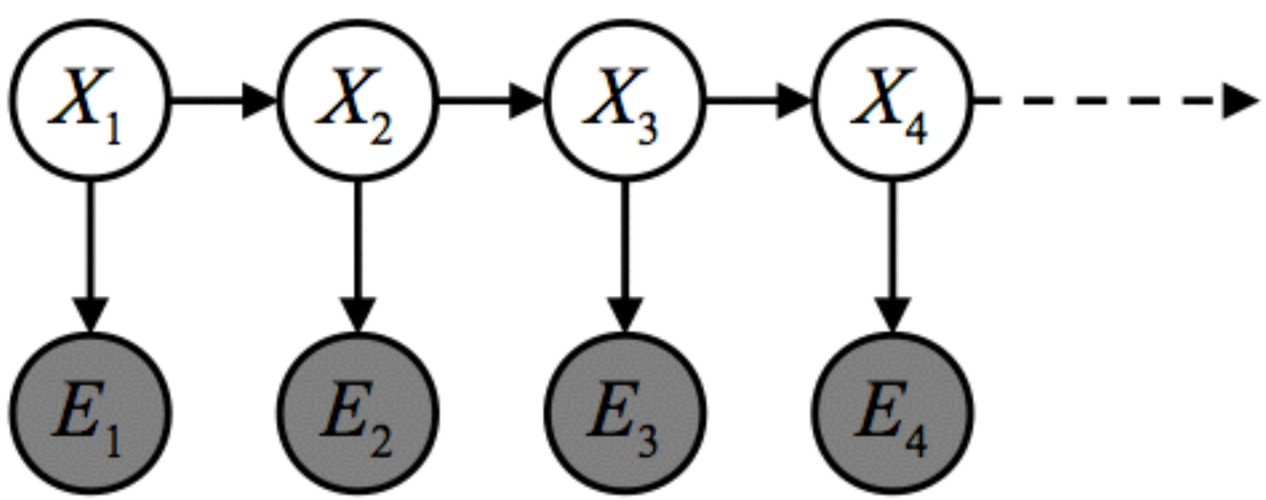
- Each web page is a state
- Initial distribution: uniform over pages
- Transitions:
  - With probability c, uniform jump to a random page (dotted lines)
  - With probability 1-c, follow a random outlink (solid lines)

- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam (but not immune)
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors

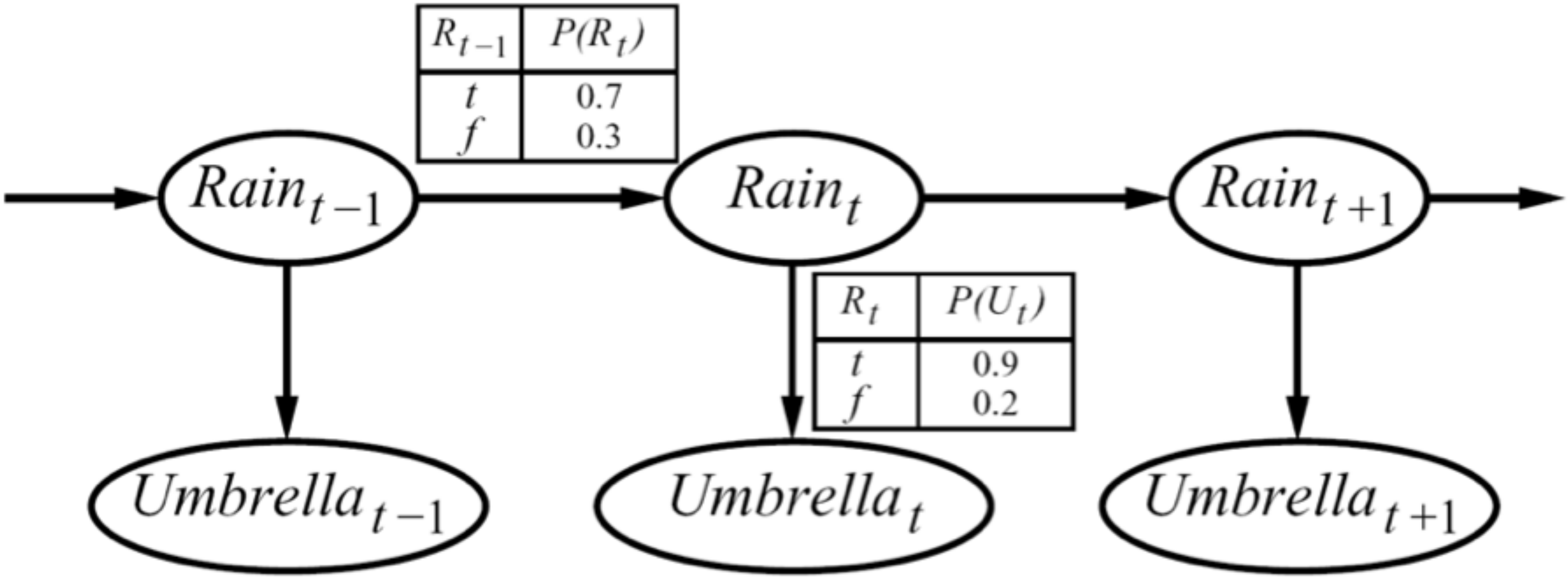
Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don't know anything anymore
  - Need observations to update your beliefs

- Hidden Markov Models (HMMs)
  - Underlying Markov chain over states S
  - You observe outputs (effects) at each time step
  - As a Bayes net:



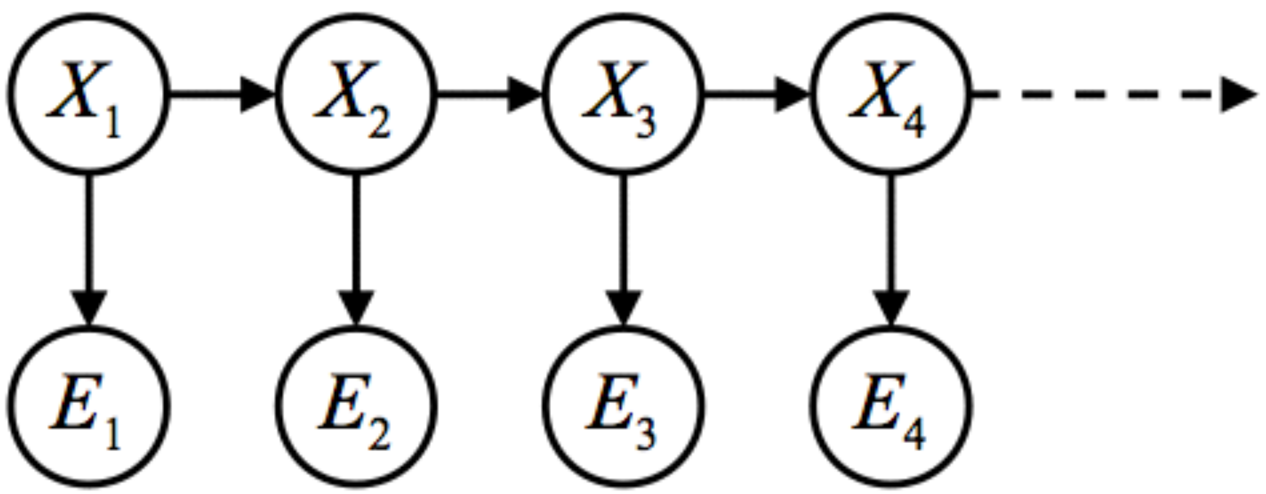
Example



- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $O(X_t \mid X_{t-1})$
  - Emissions: $P(E \mid X)$

Conditional independence

- HMMs have two important independence properties:
  - Markov hidden process, future depends on past via the present
  - Current observations independent of all else given the current state



- Quiz: does this mean that the observations are independent?
  - No, correlated by the hidden state

Real HMM examples

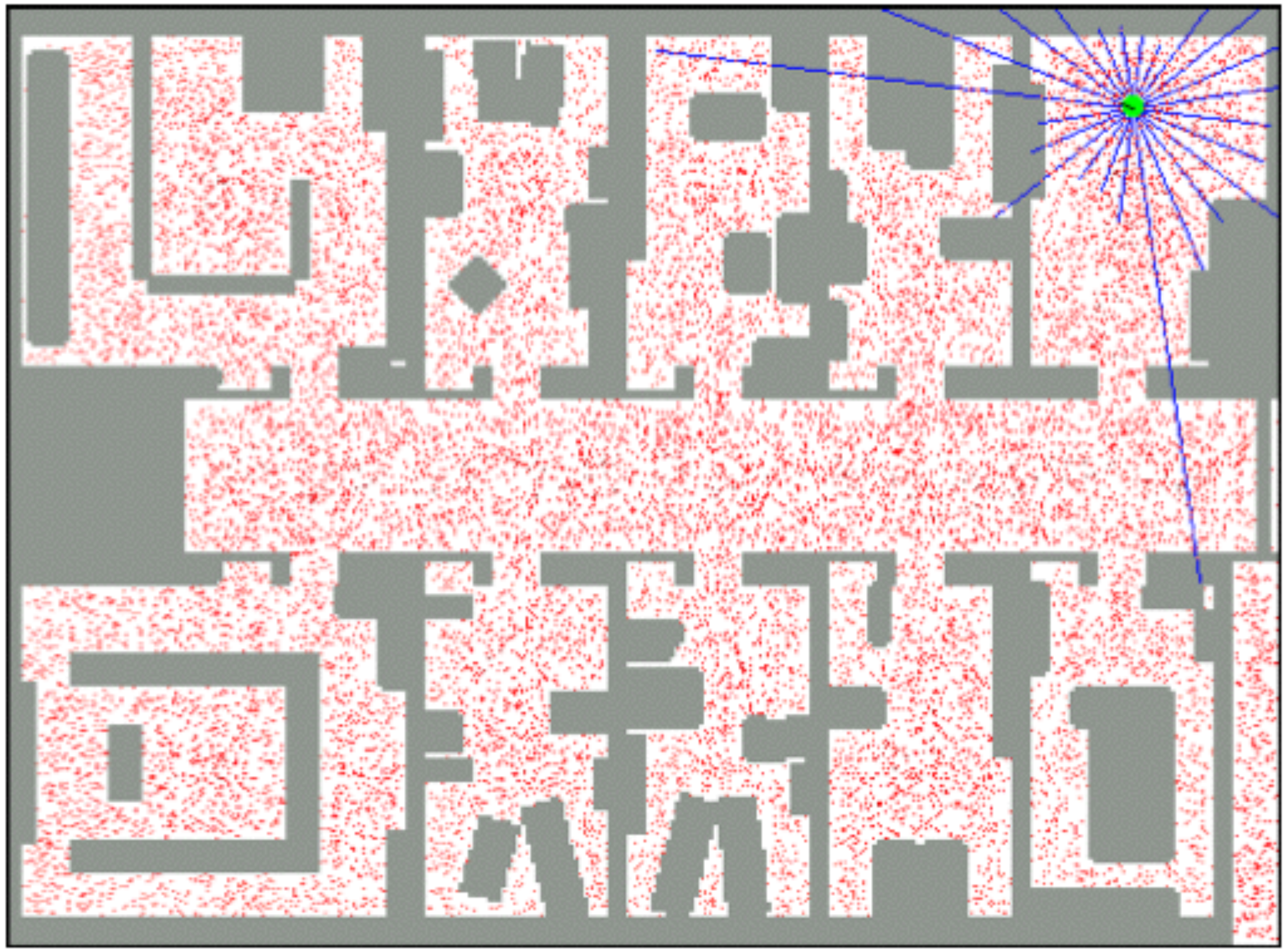
- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options

- Robot tracking HMMs:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

Filtering states

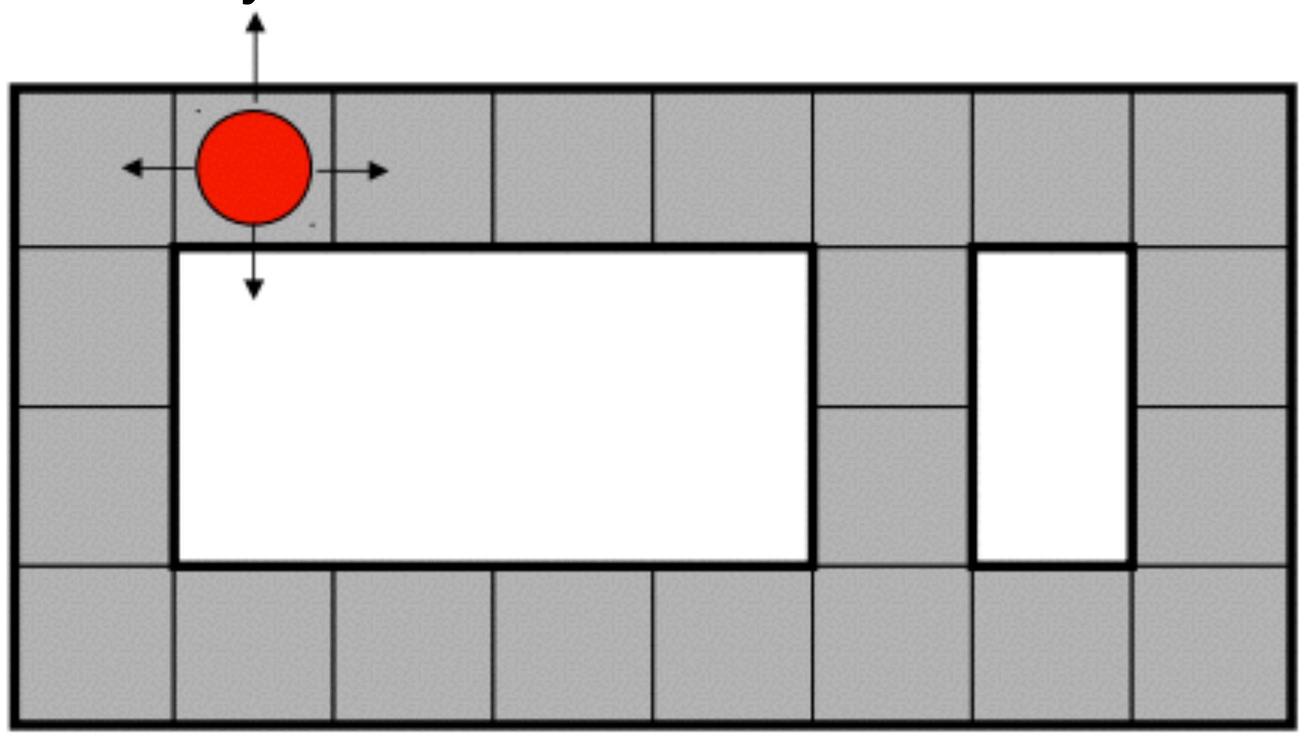
- Filtering is the task of tracking the distribution B(X) (the belief state)
- We start with B(X) in the initial setting, usually uniform
- As time passes, or we get observations we update B(X)



Example: Robot localization

Sensor model: can sense if each side has a wall or not (never more than 1 mistake)

Motion model: may not execute action with a small probability

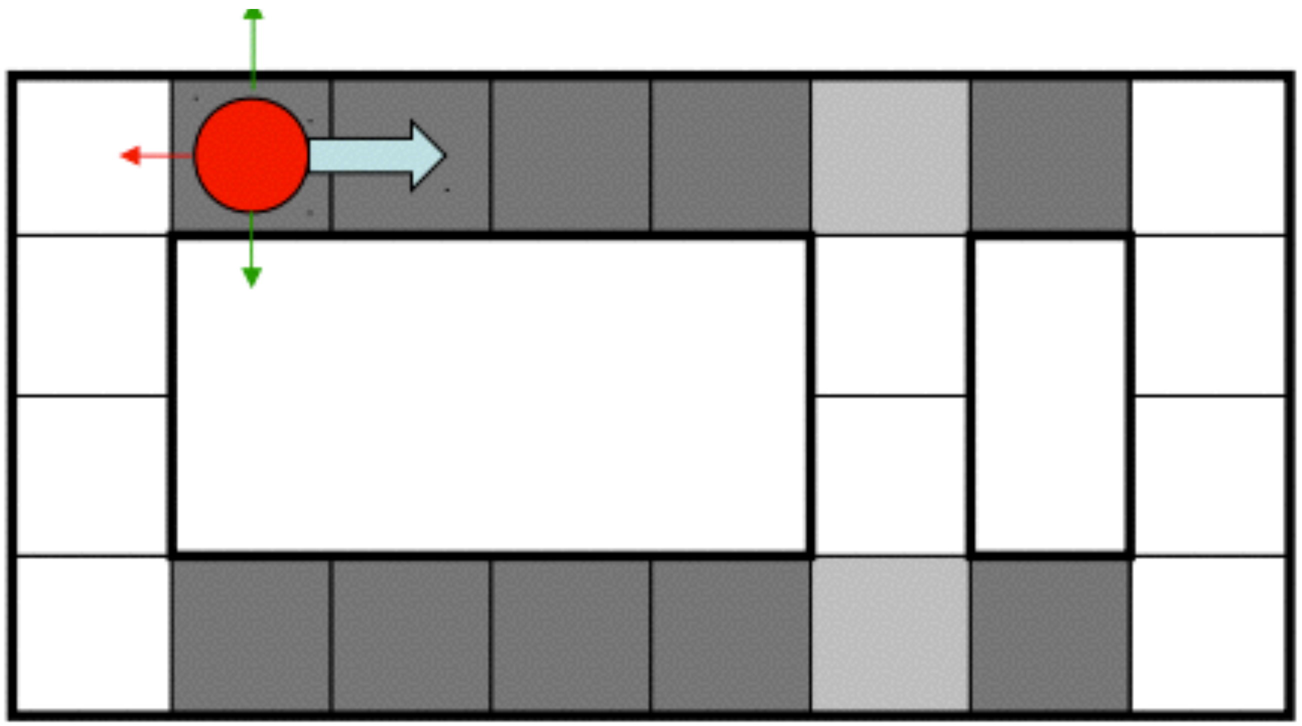


prob

t=0

Example from Michael Pfeiffer

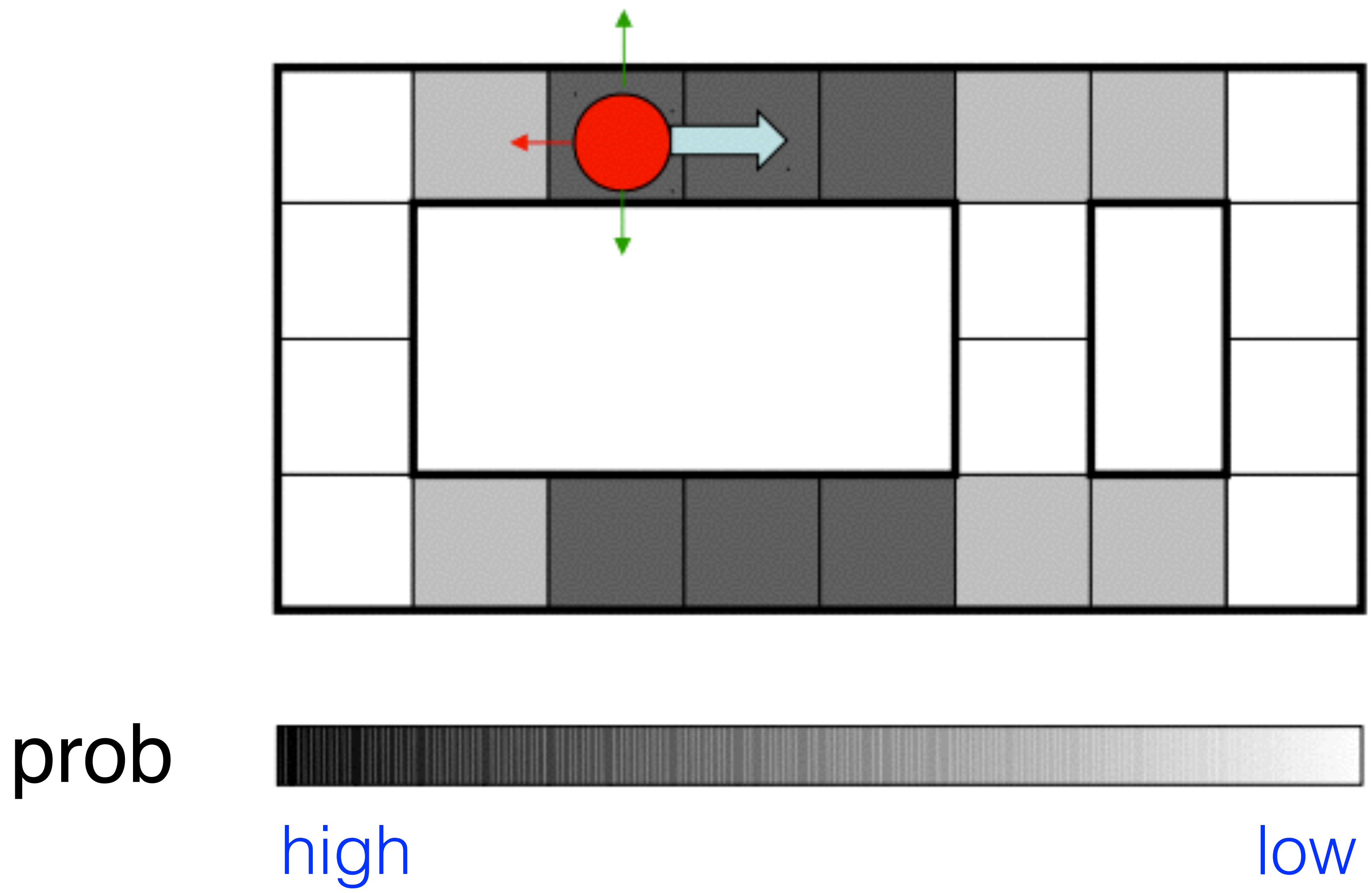
Example: Robot localization



prob

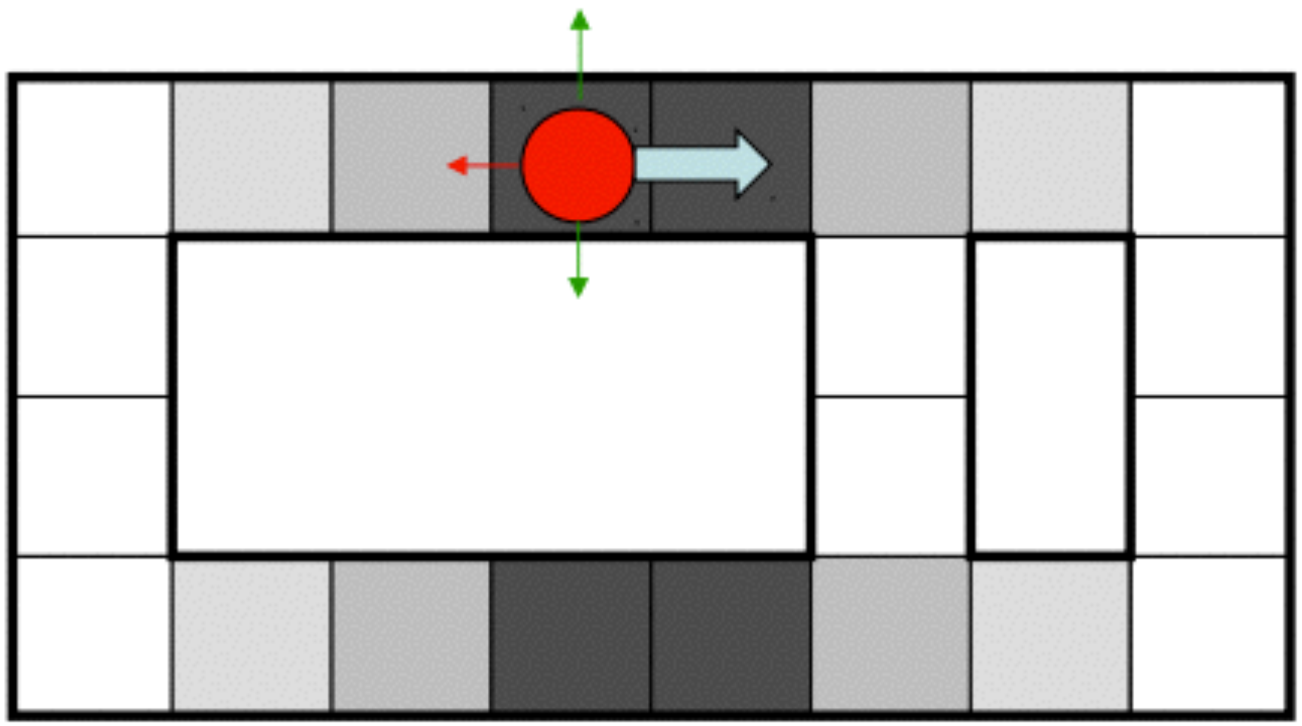
t=1

Example: Robot localization



t=2

Example: Robot localization

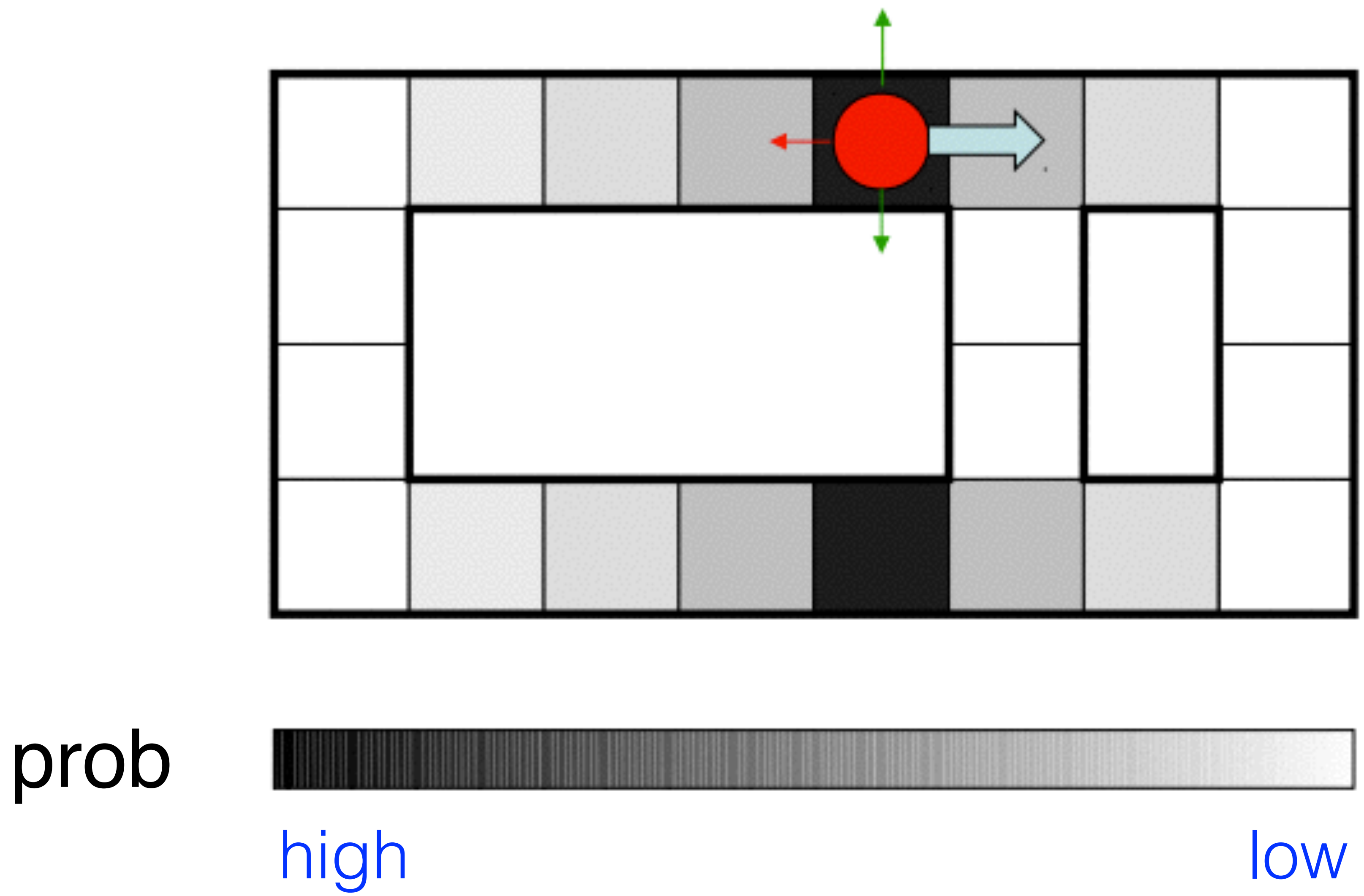


prob

high low

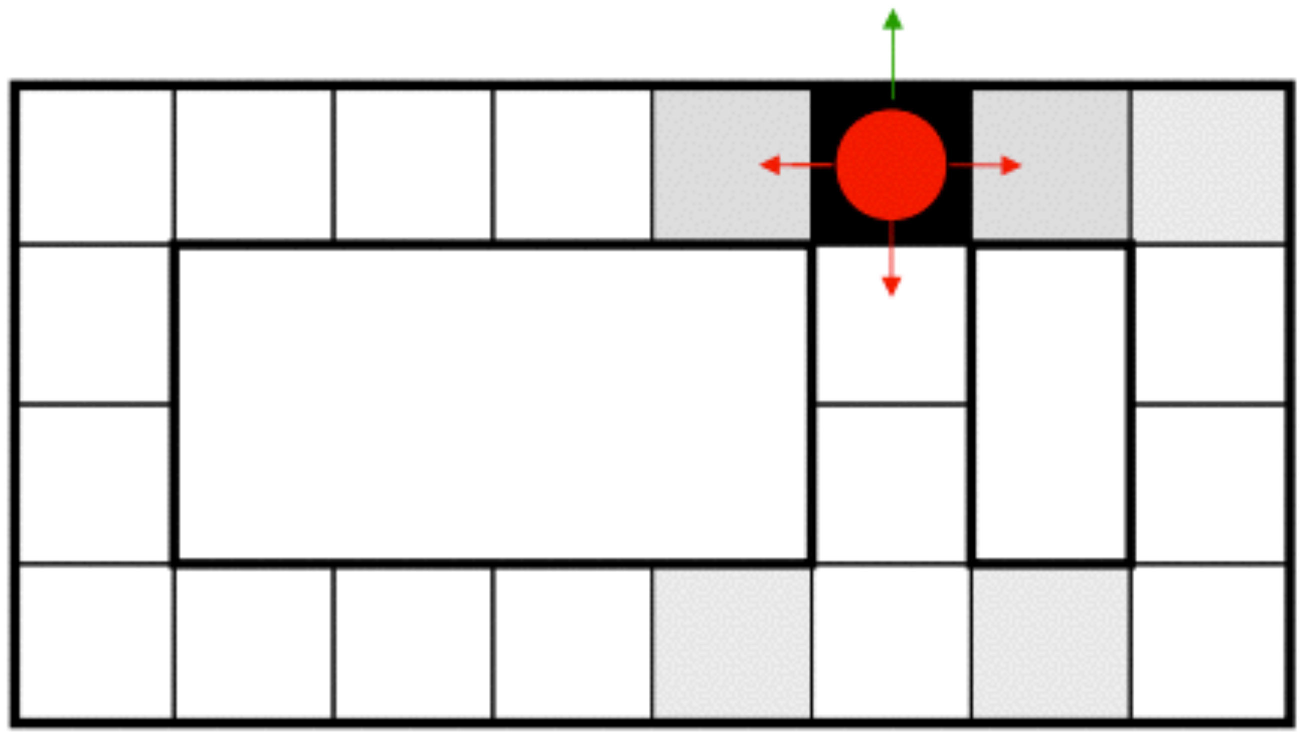
t=3

Example: Robot localization



t=4

Example: Robot localization



prob

t=5

Passage of time

- Assume we have a current belief state P(X I evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$

- Then, after one time step passes:

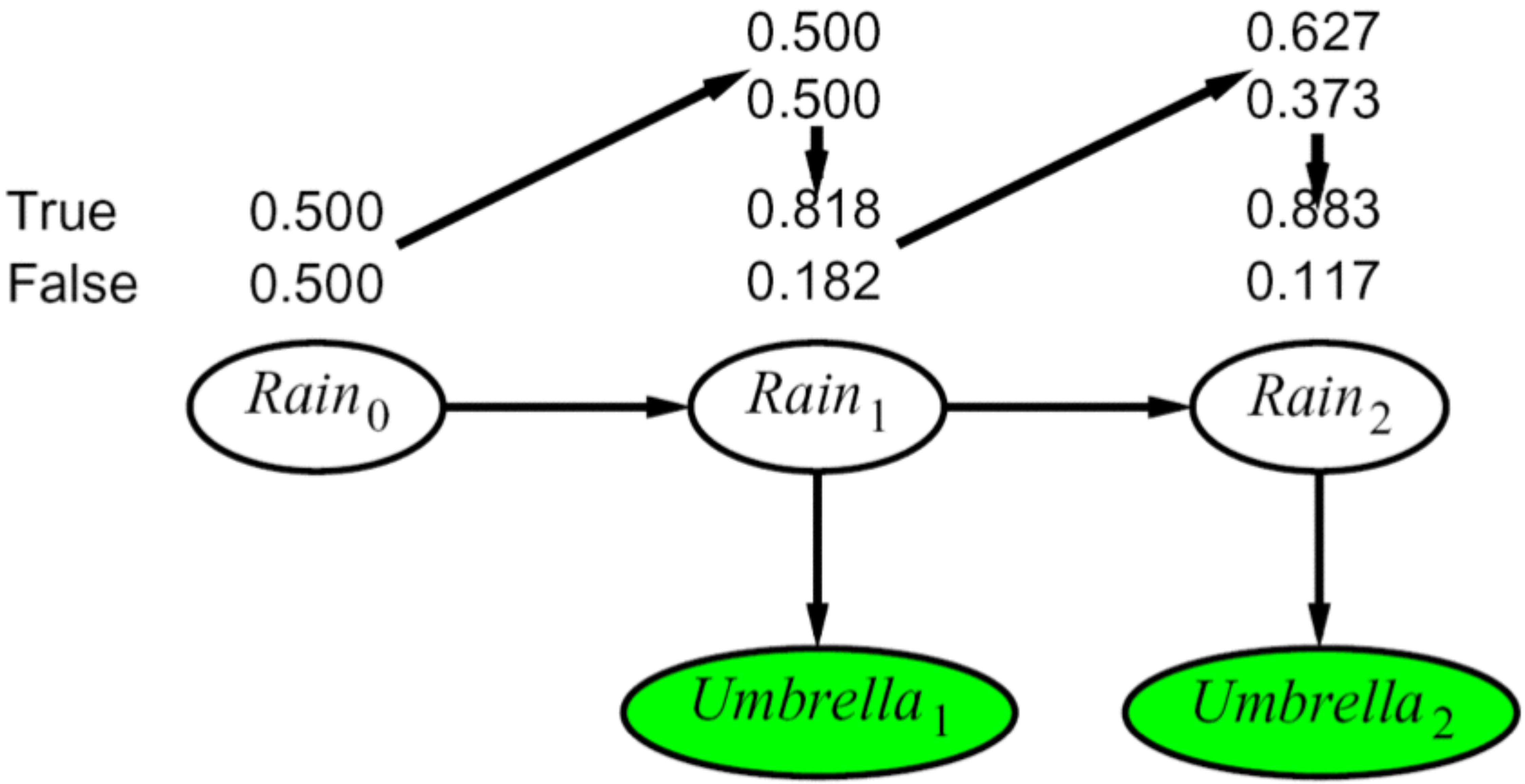
$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

- Or, compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x)B(x_t)$$

- Basic idea: beliefs get "pushed" though the transitions
  - With the "B" notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Example HMM



0.500
0.500
0.627
0.373
True
False
0.500
0.500
0.818
0.182
0.883
0.117
Rain 0
Rain 1
Rain 2
Umbrella 1
Umbrella 2

Most likely explanation

- Question: most likely sequence ending in x at time t?
  - E.g., if sun on day 4, what's the most likely sequence?
  - Intuitively: probably sun on all four days

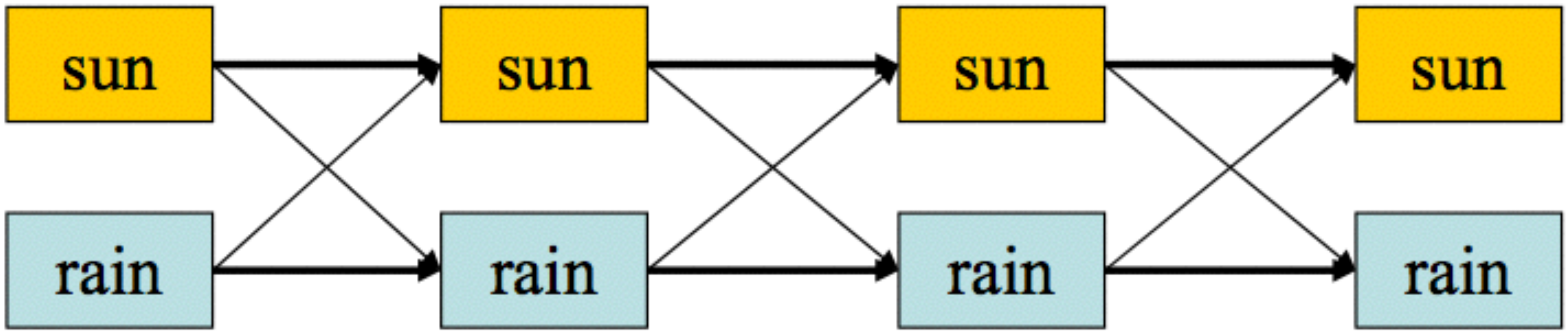
- Slow answer: enumerate and score

$$\text{most likely sequence} \leftarrow \arg \max_{x_1, \ldots, x_{t-1}} P(x_1, \ldots, x_{t-1}, \text{sun})$$

  - Complexity $O(2^{t-1})$

Mini-Viterbi algoritm

- Better answer: cached incremental updates



- Define:

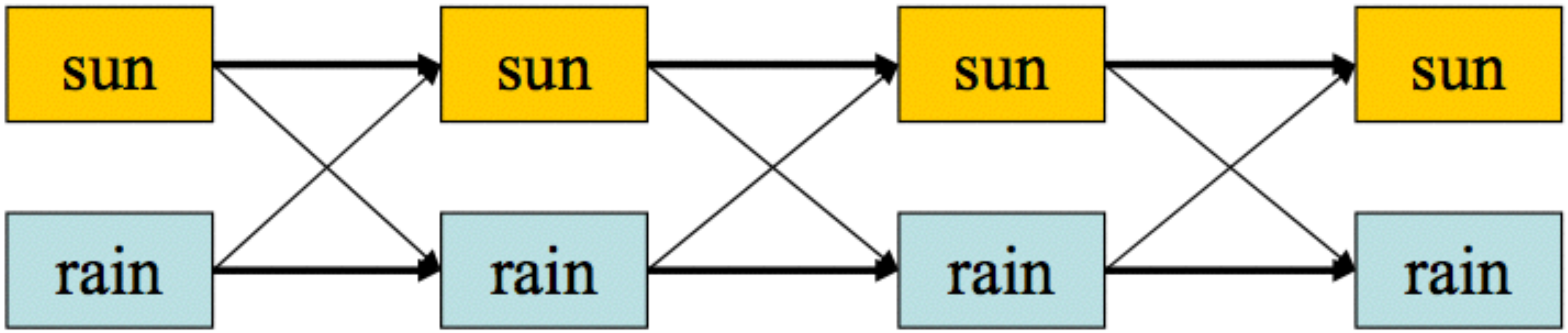
$$m_t[x] = \max_{x_{1:t-1}} P(x_{1:t-1}, x)$$

$$a_t[x] = \arg\max_{x_{1:t-1}} P(x_{1:t-1}, x)$$

- Read of the best sequence from the m and a vectors

Mini-Viterbi algoritm

- Better answer: cached incremental updates



$$
\begin{aligned}
m_t[x] &= \max_{x_{1:t-1}} P(x_{1:t-1}, x) \\
&= \max_{x_{1:t-1}} P(x_{1:t-1}) P(x|x_{t-1}) \\
&= \max_{x_{t-1}} P(x|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-2}, x_{t-1}) \\
&= \max_{x_{t-1}} P(x|x_{t-1}) m_{t-1}[x_{t-1}]
\end{aligned}
$$

$$
m_1[x] = P(x)
$$

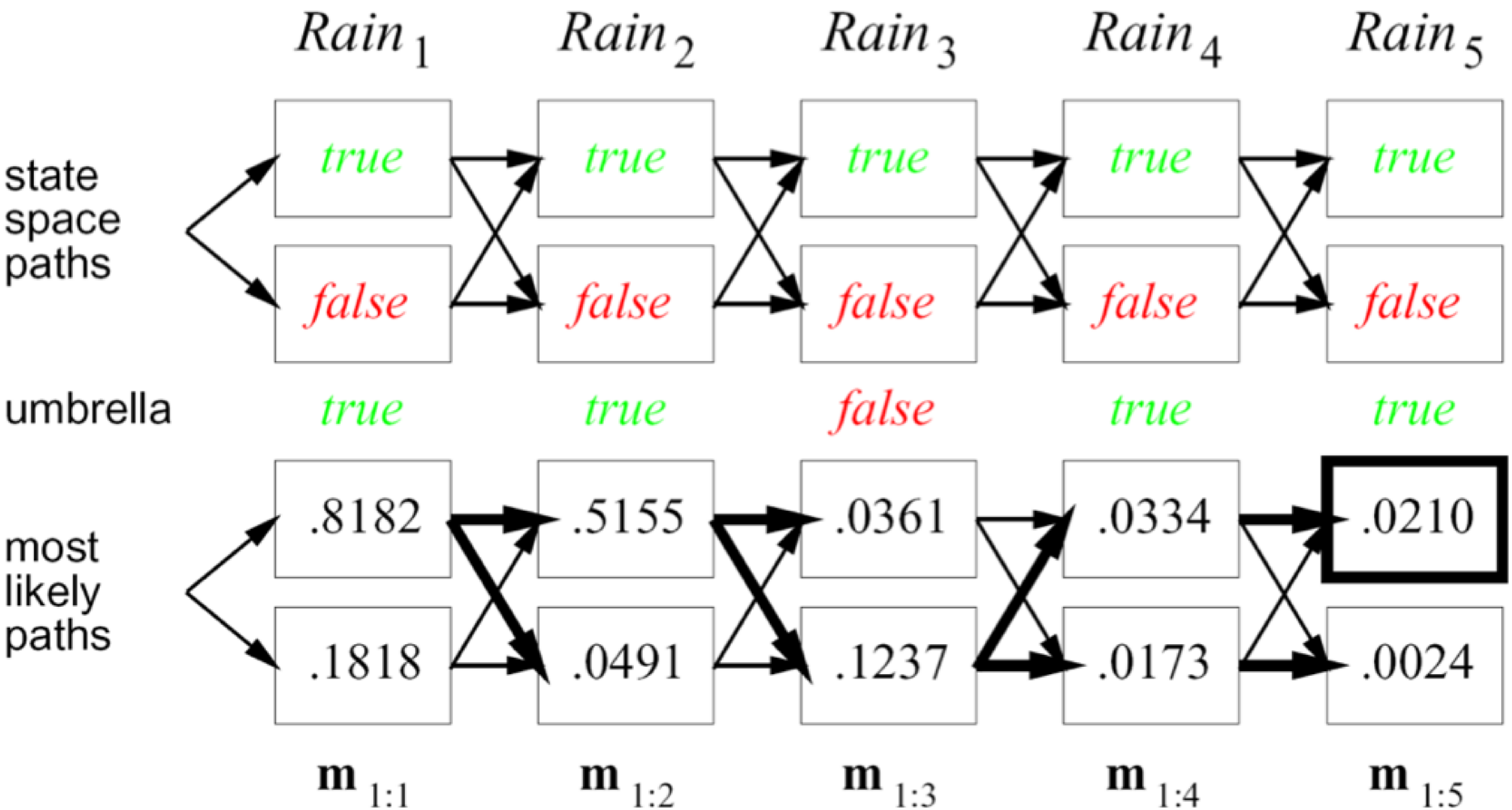
Viterbi algorithm

- Question: what is the most likely state sequence given the observations?
  - Slow answer: enumerate all possibilities
  - Better answer: cached incremental version

$$x^*_{1:t} = \arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$$

$$\begin{aligned}
m_t[x_t] &= \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t}) \\
&= \max_{x_{1:t-1}} P(x_{1:t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t) \\
&= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1}, e_{1:t-1}) \\
&= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}]
\end{aligned}$$

Example



Rain 1
Rain 2
Rain 3
Rain 4
Rain 5
state space paths
true
true
true
true
true
false
false
false
false
false
umbrella
true
true
false
true
true
most likely paths
.8182
.5155
.0361
.0334
.0210
.1818
.0491
.1237
.0173
.0024
m 1:1
m 1:2
m 1:3
m 1:4
m 1:5

Learning discrete HMMs: I

- Given sequences of training data (X^1, Y^1) (X^2, Y^2) … (X^N, Y^N) — the hidden states are known
- Maximum-likelihood parameters estimation is easy:
  - Transition probabilities:

$$P_t(a|b) = \frac{\sum_{n,t}[X_t^n = a, X_{t-1}^n = b]}{\sum_{n,t,a}[X_t^n = a, X_{t-1}^n = b]}$$

  - Emission probabilities:

$$P_e(a|b) = \frac{\sum_{n,t}[Y_t^n = a, X_t^n = b]}{\sum_{n,t,a}[Y_t^n = a, X_t^n = b]}$$

  - Initial probabilities:

$$\pi(a) = \frac{\sum_{n}[X_1^n = a]}{\sum_{n,a}[X_1^n = a]}$$

Learning discrete HMMs: II

- Given sequences of training data ($Y^1, Y^2, \ldots, Y^N$) — no hidden states
- Use EM algorithm!
  - Randomly initialize parameters of the HMM
  - E step: Compute posterior probabilities

$$q(X_t^n = a, X_{t-1}^n = b) \leftarrow p(X_t^n = a, X_{t-1}^n = b|\mathcal{D}, \Theta)$$
$$q(X_t^n = a) \leftarrow p(X_t^n = a|\mathcal{D}, \Theta)$$

dynamic programming

  - M step: Update parameters of the HMM
    - Transition probabilities: $P_t(a|b) = \dfrac{\sum_{n,t} q(X_t^n = a, X_{t-1}^n = b)}{\sum_{n,t,a} q(X_t^n = a, X_{t-1}^n = b)}$
    - Emission probabilities: $P_e(a|b) = \dfrac{\sum_{n,t} [Y_t^n = a] q(X_t^n = b)}{\sum_{n,t,a} [Y_t^n = a] q(X_t^n = b)}$
    - Initial probabilities: $\pi(a) = \dfrac{\sum_n q(X_1^n = a)}{\sum_{n,a} q(X_1^n = a)}$

Summary

- Hidden Markov Models (HMMs) for modeling sequential data
  - Parameters for a discrete HMM: transition probabilities, emission probabilities, initial state probabilities

- Inference questions —
  - What is the belief state given observations?
  - What is the most likely explanation given the observations? (Viterbi)
  - All of these can be computed using dynamic programming in **$O(S^2T)$ time** compared to brute-force enumeration that needs **$O(S^T)$** time

- Learning HMMs
  - Known hidden states — ML estimates of parameters are proportional to the counts
  - Unknown hidden states — use EM (Baum-Welch 1960)

Slides credit

- Many of the slides are adapted from those by Hal Daume III, Dan Klein, Stuart Russell or Andrew Moore
- The “sprinkler, rain, grass” figure is from Wikipedia’s discussion on Bayes networks http://en.wikipedia.org/wiki/Bayesian_network
- The robot navigation example is from Michael Pfeiffer