Perceptron

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CMPSCI 689: Machine Learning

3 February 2015

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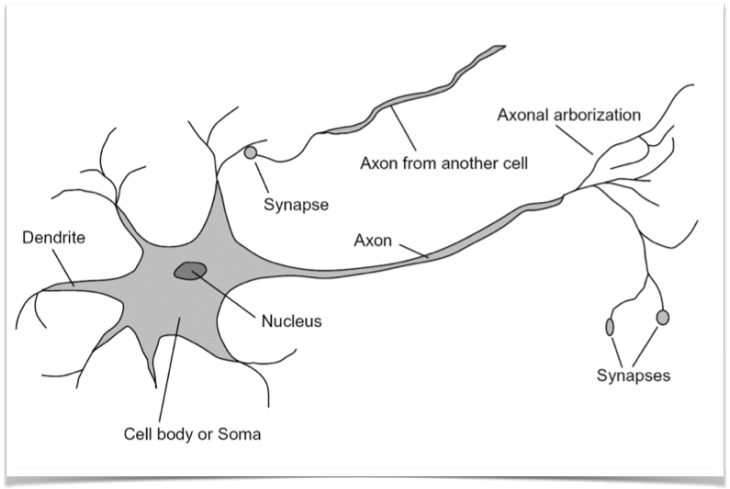
So far in the class

- Decision trees
  - Inductive bias: use a combination of small number of features
- Nearest neighbor classifier
  - Inductive bias: all features are equally good
- Perceptrons ⟵ Today
  - Inductive bias: use all features, but some more than others

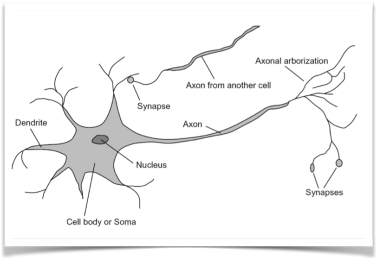
A neuron (or how our brains work)



Neuroscience 101

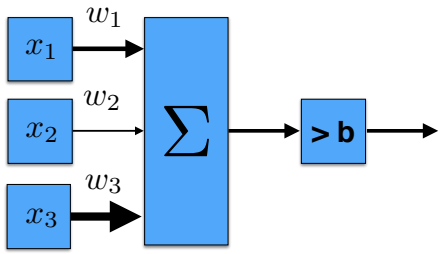
Perceptron

- Input are feature values
- Each feature has a weight
- Sum in the activation



$$\text{activation}(\mathbf{w}, \mathbf{x}) = \sum_i w_i x_i = \mathbf{w}^T \mathbf{x}$$

- If the activation is:
  - > b, output *class 1*
  - otherwise, output *class 2*



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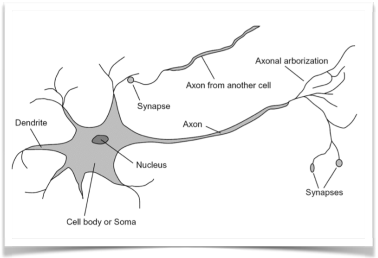

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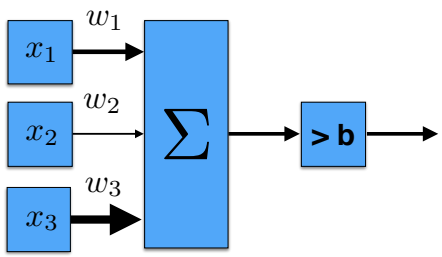
$$\mathbf{x} \rightarrow (\mathbf{x}, 1)$$
$$\mathbf{w}^T \mathbf{x} + b \rightarrow (\mathbf{w}, b)^T (\mathbf{x}, 1)$$



Example: Spam

- Imagine 3 features (spam is "positive" class):
  - free (number of occurrences of "free")
  - money (number of occurrences of "money")
  - BIAS (intercept, always has value 1)

email "free money"

$\mathbf{x}$

```
BIAS  :  1
free  :  1
money :  1
...
```

$\mathbf{w}$

```
BIAS  : -3
free  :  4
money :  2
...
```

$\mathbf{w}^T\mathbf{x}$

```
(1)(-3) +
(1)(4)  +
(1)(2)  +
...
= 3
```

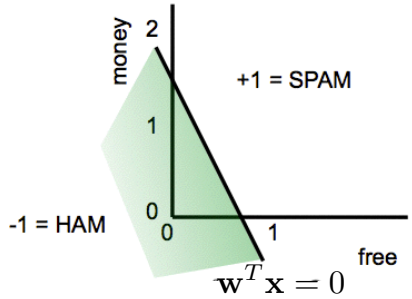
$$\mathbf{w}^T \mathbf{x} > 0 \rightarrow \text{SPAM!!}$$

Geometry of the perceptron

- In the space of feature vectors
  - examples are points (in D dimensions)
  - an weight vector is a hyperplane (a D-1 dimensional object)
  - One side corresponds to y=+1
  - Other side corresponds to y=-1
- Perceptrons are also called as linear classifiers

$\mathbf{w}$

```
BIAS  : -3
free  :  4
money :  2
...
```



Learning a perceptron

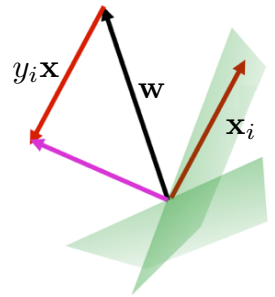
Input: training data $(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_n, y_n)$

Perceptron training algorithm [Rosenblatt 57]

- Initialize $\mathbf{w} \leftarrow [0, \ldots, 0]$
- for iter = 1,…,T
  - for i = 1,..,n
    - predict according to the current model

$$\hat{y}_i = \begin{cases} +1 & \text{if } \mathbf{w}^T \mathbf{x}_i > 0 \\ -1 & \text{if } \mathbf{w}^T \mathbf{x}_i \leq 0 \end{cases}$$

    - if $y_i = \hat{y}_i$, no change
    - else, $\mathbf{w} \leftarrow \mathbf{w} + y_i \mathbf{x}_i$



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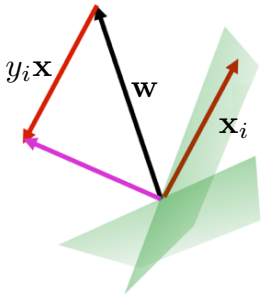


error driven, online, activations increase for +, randomize

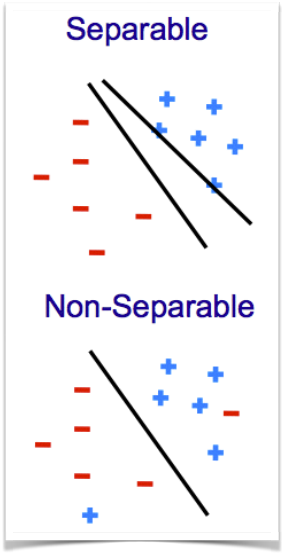
Properties of perceptrons

- **Separability:** some parameters will classify the training data perfectly
- **Convergence:** if the training data is separable then the perceptron training will eventually converge [Block 62, Novikoff 62]
- **Mistake bound:** the maximum number of mistakes is related to the margin

$$\text{assuming, } ||\mathbf{x}_i|| \leq 1$$

$$\#\text{mistakes} < \frac{1}{\delta^2}$$

$$\delta = \max_{\mathbf{w}} \min_{(\mathbf{x}_i, y_i)} \left[y_i \mathbf{w}^T \mathbf{x}_i \right] \quad \text{such that, } ||\mathbf{w}|| = 1$$



Review geometry

Proof of convergence

Let, $\hat{\mathbf{w}}$ be the separating hyperplane with margin δ

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w is getting closer

$$\begin{aligned}
\hat{\mathbf{w}}^T\mathbf{w}^{(k)} &= \hat{\mathbf{w}}^T\left(\mathbf{w}^{(k-1)} + y_i\mathbf{x}_i\right) && \text{update rule}\\
&= \hat{\mathbf{w}}^T\mathbf{w}^{(k-1)} + \hat{\mathbf{w}}^T y_i\mathbf{x}_i && \text{algebra}\\
&\geq \hat{\mathbf{w}}^T\mathbf{w}^{(k-1)} + \delta && \text{definition of margin}\\
&\geq k\delta && ||\mathbf{w}^{(k)}|| \geq k\delta
\end{aligned}$$

bound the norm

$$\begin{aligned}
||\mathbf{w}^{(k)}||^2 &= ||\mathbf{w}^{(k-1)} + y_i\mathbf{x}_i||^2 && \text{update rule}\\
&\leq ||\mathbf{w}^{(k-1)}||^2 + ||y_i\mathbf{x}_i||^2 && \text{triangle inequality}\\
&\leq ||\mathbf{w}^{(k-1)}||^2 + 1 && \text{norm}\\
&\leq k && ||\mathbf{w}^{(k)}|| \leq \sqrt{k}
\end{aligned}$$

$$k\delta \leq ||\mathbf{w}^{(k)}|| \leq \sqrt{k} \longrightarrow k \leq \frac{1}{\delta^2}$$

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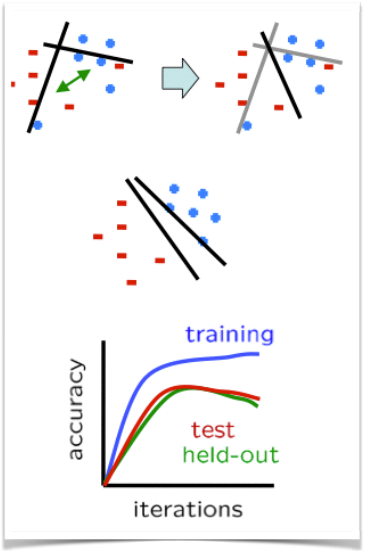
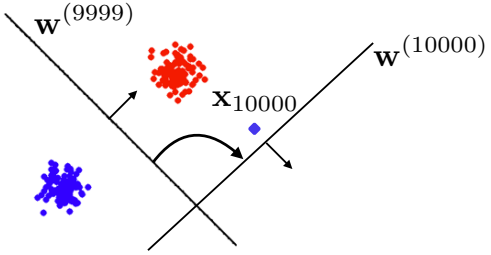
Limitations of perceptrons

- **Convergence:** if the data isn't separable, the training algorithm may not terminate
  - noise can cause this
  - some simple functions are not separable (xor)
- **Mediocre generation:** the algorithm finds a solution that "barely" separates the data
- **Overtraining:** test/validation accuracy rises and then falls
  - Overtraining is a kind of overfitting



A problem with perceptron training

- Problem: updates on later examples can take over
  - 10000 training examples
  - The algorithm learns weight vector on the first 100 examples
  - Gets the next 9899 points correct
  - Gets the 10000th point wrong, updates on the the weight vector
  - This completely ruins the weight vector (get 50% error)



- Voted and averaged perceptron (Freund and Schapire, 1999)

Voted perceptron

Key idea: remember how long each weight vector survives

- Let, $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \ldots, \mathbf{w}^{(K)}$, be the sequence of weights obtained by the perceptron learning algorithm.
- Let, $c^{(1)}, c^{(2)}, \ldots, c^{(K)}$, be the survival times for each of these.
  - a weight that gets updated immediately gets $c = 1$
  - a weight that survives another round gets $c = 2$, etc.
- Then,

$$\hat{y} = \text{sign}\left(\sum_{k=1}^{K} c^{(k)} \text{sign}\left(\mathbf{w}^{(k)T}\mathbf{x}\right)\right)$$

Voted perceptron training algorithm

Input: training data $(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_n, y_n)$

Output: list of pairs $(\mathbf{w}^{(1)}, c^{(1)}), (\mathbf{w}^{(2)}, c^{(2)}), \ldots, (\mathbf{w}^{(K)}, c^{(K)})$

- Initialize: $k = 0, c^{(1)} = 0, \mathbf{w}^{(1)} \leftarrow [0, \ldots, 0]$
- for iter = 1,…,T
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    - if $y_i = \hat{y}_i$, $c^{(k)} = c^{(k)} + 1$
    - else, $\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + y_i\mathbf{x}_i$
      $c^{(k+1)} = 1$
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Better generalization, but not very practical

Averaged perceptron

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- Let, $c^{(1)}, c^{(2)}, \ldots, c^{(K)}$, be the survival times for each of these.
  - a weight that gets updated immediately gets c = 1
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- Then,

$$\hat{y} = \text{sign}\left(\sum_{k=1}^{K} c^{(k)} \left(\mathbf{w}^{(k)T}\mathbf{x}\right)\right) = \text{sign}\left(\bar{\mathbf{w}}^T\mathbf{x}\right)$$

performs similarly, but much more practical

Averaged perceptron training algorithm

Input: training data $(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_n, y_n)$

Output: $\bar{\mathbf{w}}$

- Initialize: $c = 0, \mathbf{w} = [0, \ldots, 0], \bar{\mathbf{w}} = [0, \ldots, 0]$
- for iter = 1,...,T
  - for i = 1,..,n
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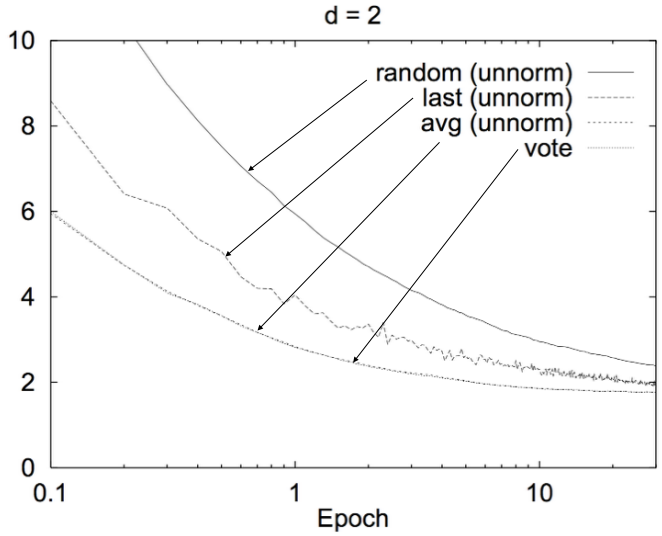
    - if $y_i = \hat{y}_i$, $c = c + 1$
    - else, $\bar{\mathbf{w}} \leftarrow \bar{\mathbf{w}} + c\mathbf{w}$
      $\mathbf{w} \leftarrow \mathbf{w} + y_i\mathbf{x}_i$
      $c = 1$
- Return $\bar{\mathbf{w}} + c\mathbf{w}$

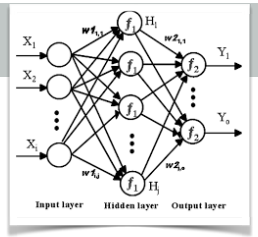
update average

Comparison of perceptron variants

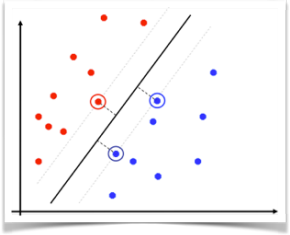
MNIST dataset (Figure from Freund and Schapire 1999)



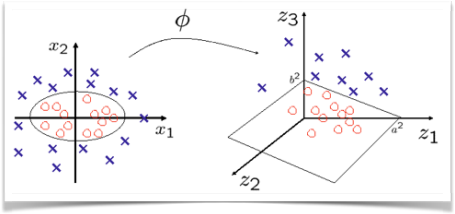
Improving perceptrons



- **Multilayer perceptrons:** to learn non-linear functions of the input (neural networks)
- **Separators with good margins:** improves generalization ability of the classifier (support vector machines)



- **Feature-mapping:** to learn non-linear functions of the input using a perceptron
  - we will learn do this efficiently using kernels



$$\phi : (x_1, x_2) \rightarrow (x_1^2, \sqrt{2}x_1x_2, x_2^2)$$

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \rightarrow \frac{z_1}{a^2} + \frac{z_3}{b^2} = 1$$

Slides credit

- Some slides adapted from Dan Klein at UC Berkeley and CIML book by Hal Daume
- Figure comparing various perceptrons are from Freund and Schapire