Simple Graph Algorithms in the Semi-streaming Model & Map Reduce

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Graph Stream

- Consider a stream of m edges

$$\langle e_1, e_2, \ldots \quad \ldots, e_m \rangle$$

  defining a graph G with nodes $V = [n]$ and $E = \{e_1, \ldots, e_m\}$
- Massive graphs include social networks, web graph, call graphs, etc.
- What can we compute about G in $o(m)$ space?
- Focus on *semi-streaming* space restriction of $O(n \cdot \text{polylog}\, n)$ bits.

Connectivity

- *Goal:* Compute the number of connected components.
- *Algorithm:* Maintain a spanning forest F
  - $F \leftarrow \emptyset$
  - For each edge (u, v), if u and v aren't connected in F,

$$F \leftarrow F \cup \{(u, v)\}$$

- *Analysis:*
  - F has the same number of connected components as G
  - F has at most $n - 1$ edges.
- *Thm:* Can count connected components in $O(n \log n)$ space.

K-connectivity

- *Goal:* Check if all cuts are of size at least k.
- *Algorithm:* Maintain k forests $F_1, \ldots, F_k$
  - $F_1, \ldots, F_k \leftarrow \emptyset$
  - For each edge (u, v), find smallest $i \leq k$ such that u and v aren't connected in F_i,
    $$F_i \leftarrow F_i \cup \{(u, v)\}$$
    If no such i exists, ignore edge.
- *Analysis:*
  - Each F_i has at most $n - 1$ edges so total edges is $O(nk)$
  - *Lemma:* Min-Cut$(V, E) < k$ iff Min-Cut$(V, F_1 \cup \ldots \cup F_k) < k$
- *Thm:* Can check k-connectivity in $O(kn \log n)$ space.

Proof of Lemma

- Let $H = (V, F_1 \cup \ldots \cup F_k)$ and let $(S, V \setminus S)$ be an arbitrary cut.
- Since H is a subgraph:

$$|E_G(S)| \geq |E_H(S)|$$

  where $E_H(S)$ and $E_G(S)$ are the edges across the cut in H and G
- Suppose there exists $(u, v) \in E_G(S)$ but $(u, v) \notin F_1 \cup \ldots \cup F_k$. Then (u, v) must be connected in each F_i. Since F_i are disjoint,

$$|E_H(S)| \geq \min(|E_G(S)|, k)$$

Minimum Spanning Forest

- Goal: Obtain the minimum spanning forest
- Algorithm: Maintain a spanning forest F
  - *Initialize F ← Φ*
  - *Edge (u,v) arrives*
    - *If u and v not connected in F, insert (u,v) in F*
    - *If u and v are connected in F then include (u,v) and find the cycle containing it— remove the edge with minimum weight in that cycle*
- Analysis
  - F is a forest
  - If an edge (u,v) is not in F then (u,v) must be the heaviest weight edge in some cycle in G
- Thm: Can maintain minimum spanning tree in O(nlog(n)) space

Minimum Spanning Tree in Map Reduce

- Distribute edges randomly to machines. Compute MST on local edges—Combine and Repeat!
- Analysis:
  - Correctness:
    - Use the fact that if an edge is discarded by a machine then it must be the heaviest weight edge in some cycle in a subgraph→ heaviest weight edge in that same cycle in the original graph
    - Hence combine and repeat is a valid policy
  - Complexity
    - Number of rounds required is at most $\left\lceil \frac{c}{\epsilon} \right\rceil$
    - Number of edges before the 1st round $m_1=n^{1+c}$
    - Number of edges before the 2nd round $m_2=(n-1)*n^{c-\varepsilon}=n^{1+c-\varepsilon}$ and so on

Minimum Spanning Tree in Map Reduce

- Can we partition the vertices?
  - A more complex algorithm by partitioning the vertices exist with nearly same complexity
  - Works under the same principle of combine & repeat

Graph Streams

- Sampling Edges
  - Connectivity, MST, Spanners, Sparsifiers, maximum density estimation….
- Sampling Vertices
  - Estimating graph statistic like number of paths of length two/three etc.

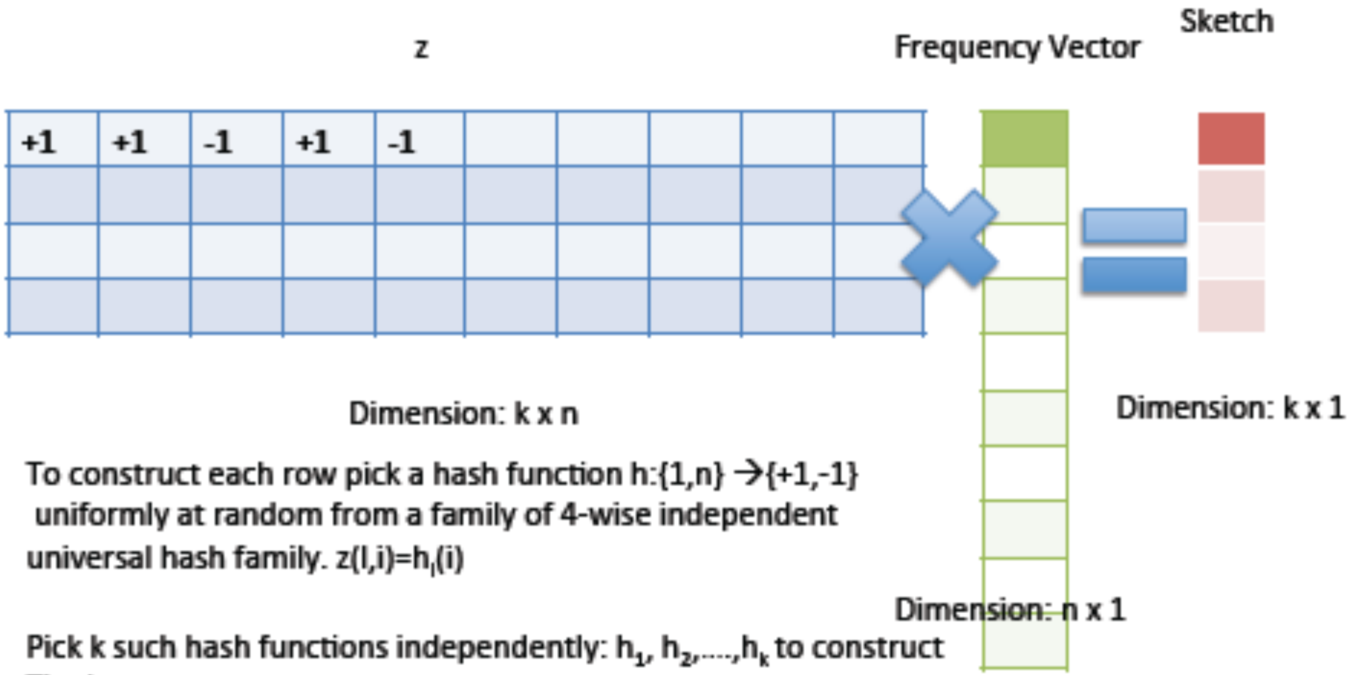
Linear Sketch

- Random linear projection $M: R^n \rightarrow R^k$ that preserves properties of any $v \in R^n$ with high probability where $k \ll n$.

$$\begin{pmatrix} & M & \end{pmatrix} \begin{pmatrix} \\ v \\ \\ \end{pmatrix} = \begin{pmatrix} Mv \end{pmatrix} \longrightarrow \text{answer}$$

- *Many Results:* Estimating norms, entropy, support size, quantiles, heavy hitters, fitting histograms and polynomials, ...
- *Rich Theory:* Related to compressed sensing and sparse recovery, dimensionality reduction and metric embeddings, ...

Linear Sketch for F_2



To construct each row pick a hash function h:{1,n} → {+1,-1} uniformly at random from a family of 4-wise independent universal hash family. $z(l,i)=h_l(i)$

Pick k such hash functions independently: $h_1, h_2, \ldots, h_k$ to construct The k rows.

Advantages of Linear Sketch

- Can handle deletion in streams
- Allows for distributed computing

- Exercise: Implement a MapReduce algorithm for computing F_2 where the stream is decomposed into k substeams and sent to k different machines initially.

- Similarly there exists linear sketches for graphs to handle deletion of edges.

Sliding/Decaying Window Model

- Only the last W items matter
  - Can you extend the algorithms for Count Min sketch and F_2 estimation in the sliding window model?
- Decaying window model
  - No fixed window size but older items have less importance
    - Can you extend the algorithms for Count Min sketch and F_2 estimation in the sliding window model?