Hashing

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Random Load Balancing

Suppose a content delivery network like YouTube receives a million content requests per minute. Each request needs to be served from one of the 1000 servers. How should one distribute the load so that no server is overloaded.

– Assign each request to a random server.

Random Load Balancing

- Let there be “n” requests and “k” servers
- Consider server “i”
- Define an indicator random variable X_j which will be 1 if request j is assigned to server i and 0 otherwise.
- Load on machine i: $$X = \sum_{j=1}^{n} X_j$$
- $E[X] = \sum_{j=1}^{n} E[X_j] = \frac{n}{k}$ [Apply the Chernoff Bound]

Random Load Balancing

- Applying the Chernoff Bound we get

$$Prob\left(X \geq \frac{n}{k} + 3\sqrt{\frac{n \ln k}{k}}\right)$$

$$= Prob\left(X \geq \frac{n}{k}(1 + 3\sqrt{\frac{\ln k}{n/k}})\right)$$

$$\leq e^{-\frac{\frac{n}{k}\frac{9 \ln k}{n/k}}{3}} = \frac{1}{k^3}$$

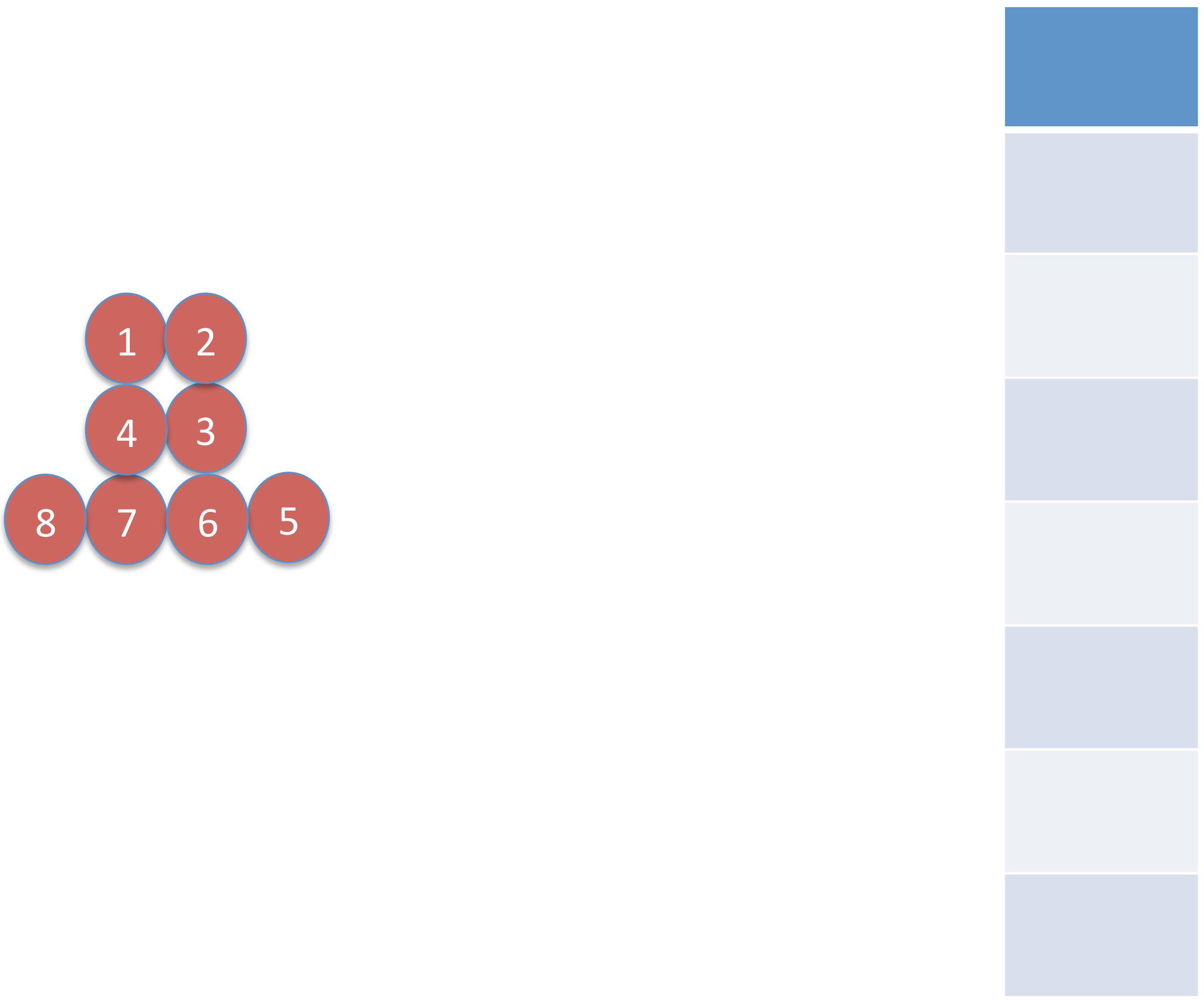
Random Load Balancing

- Apply Union Bound

- Prob(there exists at least one server which is overloaded) $\leq \frac{1}{k^2}$

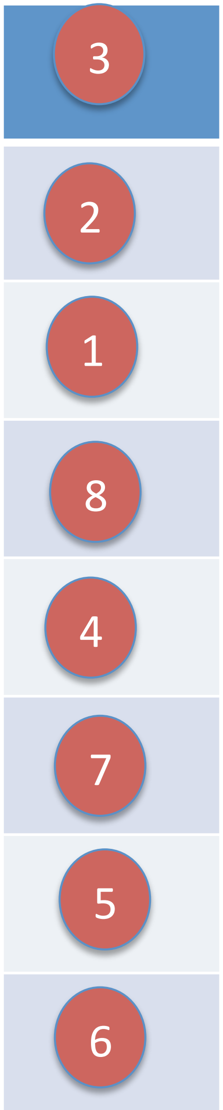
Balls in Bin

- Suppose you throw m balls into n bins randomly and uniformly, m >=n.
- If m=n, then the expected number of balls in each bin is just one.
- **EXTRA CREDIT** [20]: Show that the maximum number of balls in any bin is $O(\frac{\ln n}{\ln \ln n})$ with probability $1 - \frac{1}{n}$.



Throw the balls uniformly at random in the bins

Which bin does Ball 7 occupy?



Throw the balls uniformly at random in the bins

Hash Function

- A hash function from a universe U=[0,1,2,..,m-1] into a range [0,1,2,..,n-1] can be thought of as a way of placing m balls into n bins.
- We have a family of hash functions H and we choose one function from this family uniformly at random.

Perfectly Random Hash Functions

- A family of hash functions H is perfectly random if the following holds

$$Prob_{h \sim H}(h(x) = y) = \frac{1}{n} \text{ for all } x \in [1, m] \text{ and } y \in [1, n]$$

$$Prob_{h \sim H}(h(x_1) = y_1 \cap h(x_2) = y_2 \cap ....h(x_k) = y_k) = \frac{1}{n^k} \text{ for all } k, x_i\text{s and } y_j\text{s}$$

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Finding hash functions that are perfectly random is difficult in practice. Also, storage and time required to compute such hash functions become prohibitive.

Strongly Universal Hash Family

- Let U be a universe with $U = \{0, 1, 2, .., m-1\}$ and let $V = \{0, 1, 2, .., n-1\}$. A family of hash functions $\mathcal{H}$ is said to be strongly k-universal if for any element $x_1, x_2, .., x_k \in U$, any values $y_1, y_2, .., y_k \in U$, and a hash function h chosen uniformly at random from $\mathcal{H}$, we have

$$\boxed{Prob(h(x_1) = y_1 \cap h(x_2) = y_2 \cap .... \cap h(x_k) = y_k) = \frac{1}{n^k}}$$

2-Universal Hash Family

Example 1 (A 2-strongly Universal Family of Hash Functions). *Consider the family of hash functions obtained by choosing a prime number $p \geq m$, letting*

$$h_{a,b}(x) = ((ax + b) \ mod \ p) \ mod \ n$$

and then taking the family

$$\mathcal{H} = \{h_{a,b} \mid 1 \leq a \leq p - 1, 0 \leq b \leq p\}$$

Note that it is important that a cannot be 0

Applications

- Password Checker
  - Stores a dictionary of unacceptable password
  - When a user tries to set a password, it is first checked with this dictionary

- Possible solutions
  - Store the passwords in alphabetical order
    - Binary search
  - Use Hash function and store it in a hashtable

Applications

- Password Checker
  - Stores a dictionary of unacceptable password
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  - Store the passwords in alphabetical order
    - Binary search O(log m)
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    - O(1) expected time

Applications

- Spam Detection
  - Prevents sending spam emails to the inbox by keeping a dictionary of acceptable email ids.
  - When an email arrives check if it belongs to the list and accept if it does.

- Possible Solutions
  - Store the email ids in alphabetical order
    - Binary search O(log m)
  - Use Hash function and store it in a hash table
    - O(1) expected time

Limitations

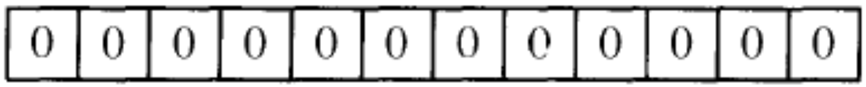
- Worst case time could be large
- Space usage may not be ideal

Bloom Filter

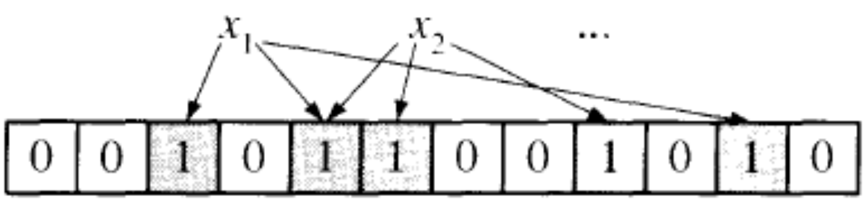
- Hash table has size "m" but now it stores only bits
  - Saves space
- Worst case time to search is O(1)
  - Saves time

Bloom Filter

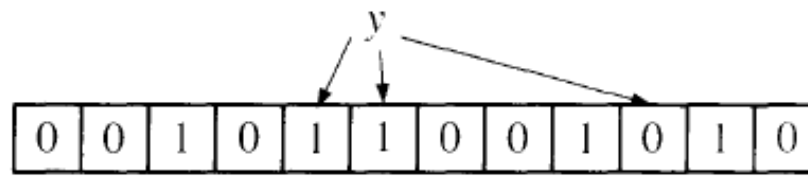
Start with an array of 0s.



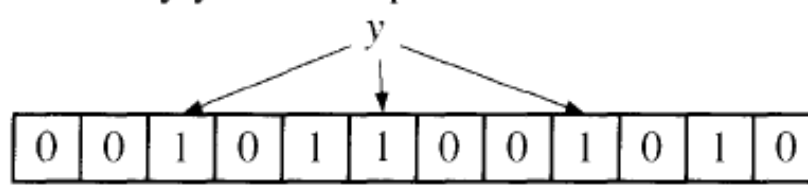
Each element of S is hashed k times; each hash gives an array location to set to 1.



To check if y is in S, check the k hash locations. If a 0 appears, y is not in S.



If only 1s appear, conclude that y is in S. This may yield false positives.



What are the false positive and false negative rates?