Probability Review

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Probability Review

- Sample Space: The set Σ of all possible outcomes of a random experiment

Example: Toss a coin twice

$$\Sigma = \{HH, HT, TH, TT\}$$

Example: Throw a dice once

$$\Sigma = \{1, 2, 3, 4, 5, 6\}$$

Probability Review

- Event: Subset of Sample Space

Example: Both the coins have the same output

$$E = (HH, TT)$$

Example: Throw of a dice returns an even number

$$E = (2, 4, 6)$$

Probability Review

- Union Bound: Will be using it again and again

$$Prob(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} Prob(E_i)$$

- When the events are all mutually disjoint

$$Prob(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} Prob(E_i)$$

Probability Review

- Random Variable: Function mapping Σ to $\mathbb{R}$
- Example: If a coin toss returns head you get $5 else you lose $5. Define a random variable X as follows
  - X(H)=+5 and X(T)=-5
  - X=+5 if coin toss returns head and -5 otherwise
- Indicator Random Variable: Takes value either 0 or 1.
- We will be dealing with discrete random variables in this course
- Two random variables are independent if

$$\forall a, b \; Prob(X = a \cap Y = b) = Prob(X = a) * Prob(Y = b)$$

Probability Review

- Expectation: Expectation of a discrete random variable X is defined as

$$E[X] = \sum_a a Prob(X = a)$$

- Example: X(H)=+5, X(T)=-5 then E[X]=0 if the coin is fair.
- Linearity of Expectation

$$E[X + Y] = E[X] + E[Y]$$

Probability Review

- Variance: variance of a random variable X is defined as

$$\begin{aligned} Var(X) &= E[(X - E(X))^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

- Linearity of Variance: holds only when the random variables are pairwise independent

Probability Review

- EXERCISE

- Suppose X and Y are two random variables that are pairwise independent. Show
  - $E[XY]=E[X]E[Y]$
  - $Var[X+Y]=Var[X]+Var[Y]$

Concentration Inequalities

- Measure the extent to which a random variable can differ from its expected value.

- Suppose you toss a fair coin 100 times.The expected number of heads is 50.

- What is the probability that you get more than 75 heads?
- What is the probability that you get less than 25 heads?

- Concentration Inequalities provide a bound on these unexpected behaviors.

Markov Inequality

- For any non-negative random variable X

$$Prob(X \geq t) \leq \frac{E[X]}{t}$$

- Let t=aE[X] to get

$$Prob(X \geq aE[X]) \leq \frac{1}{a}$$

- What is the probability of getting more than 75 heads when a fair coin is tossed 100 times?

$$Prob(X \geq 75) = Prob(X \geq \frac{3}{2}50) \leq \frac{2}{3}$$

Chebyshev's Inequality

Holds for any random variable

$$Prob(|X - E[X]| \geq t) \leq \frac{Var(X)}{t^2}$$

Let t=aE[X] then we get

$$Prob(|X - E[X]| \geq aE[X]) \leq \frac{Var(X)}{a^2 E[X]^2}$$

What is the probability of getting more than 75 heads or less than 25 heads in 100 tosses of a fair coin?