# COMPSCI 240: Reasoning Under Uncertainty

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Lecture 15: Normal/Gaussian Random Variables

# Normal (Gaussian) Random Variables

• A **normal random variable** *X* is a continuous random variable with probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

- The range of this random variable is  $\mathcal{X} = (-\infty, \infty)$ .
- The parameter  $\sigma$  must be strictly greater than 0.
- The parameter  $\mu$  can be any real value.
- Often also abbreviated as  $X \sim \mathcal{N}(\mu, \sigma^2)$ .
- Normal random variables have extremely important theoretical properties and are a default choice for modeling problems involving continuous measurements.

# Why Study Normal RV?

- Normal RV plays an important role in a broad disciplines, including but not limited to computer science, engineering, physical, and statistical context.
- Few examples related to computer science include linear regression and Gaussian Process.
- The normal random variable is a convenient tool to approximate various types of phenomena (or observations), which allows us to derive mathematically tractable solutions.
- The key fact is that the sum of a large number of independent and identically distributed (not necessarily normal) random variables has an approximately normal behavior (Central Limit Theorem).

# Normal (Gaussian) Random Variables



### Mean, Variance, and CDF

• The mean and variance can be calculated to be

$$E[X] = \mu$$
 and  $var(X) = \sigma^2$ 

• Its CDF is defined as

$$F_X(x) = P(X \le x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

• The **probability mass** of an interval [*a*, *b*] is the definite integral:

$$P(a < X < b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$
$$= F_X(b) - F_X(a)$$

# Normal (Gaussian) Random Variables



# Preserving the Normality (or Gaussianity)

• Let X be a normal random variable with mean  $\mu$  and variance  $\sigma^2$ , and if  $a \neq 0$  and b are scalars, then a random variable

$$Y=aX+b,$$

is also a normal random variable with

$$E[Y] = a\mu + b$$
 and  $var(Y) = a^2\sigma^2$ .

#### The Standard Normal Random Variable

- A normal random variable X with E[X] = 0 and var(X) = 1 is said to be a **standard normal random variable**.
- Its PDF can be simplified as

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2}$$



### The Standard Normal Random Variable

• Its CDF can be defined as

$$\Phi(x) = P(X \le x) = P(X < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$$

# The Standard Normal Table

0	.01	.02	.03	.04	.05	.06	.07	.08	.09
.5	.50399	.50798	.51197	.51595	.51994	.52392	.5279	.53188	.53586
.53983	.5438	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
.61791	.62172	.62552	.6293	.63307	.63683	.64058	.64431	.64803	.65173
.65542	.6591	.66276	.6664	.67003	.67364	.67724	.68082	.68439	.68793
.69146	.69497	.69847	.70194	.7054	.70884	.71226	.71566	.71904	.7224
.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.7549
.75804	.76115	.76424	.7673	.77035	.77337	.77637	.77935	.7823	.78524
.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
.86433	.8665	.86864	.87076	.87286	.87493	.87698	.879	.881	.88298
.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
.9032	.9049	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
.91924	.92073	.9222	.92364	.92507	.92647	.92785	.92922	.93056	.93189
.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
.9452	.9463	.94738	.94845	.9495	.95053	.95154	.95254	.95352	.95449
.95543	.95637	.95728	.95818	.95907	.95994	.9608	.96164	.96246	.96327
.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
.97128	.97193	.97257	.9732	.97381	.97441	.975	.97558	.97615	.9767
.97725	.97778	.97831	.97882	.97932	.97982	.9803	.98077	.98124	.98169
.98214	.98257	.983	.98341	.98382	.98422	.98461	.985	.98537	.98574
.9861	.98645	.98679	.98713	.98745	.98778	.98809	.9884	.9887	.98899
.98928	.98956	.98983	.9901	.99036	.99061	.99086	.99111	.99134	.99158
.9918	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
.99379	.99396	.99413	.9943	.99446	.99461	.99477	.99492	.99506	.9952
.99534	.99547	.9956	.99573	.99585	.99598	.99609	.99621	.99632	.99643
.99653	.99664	.99674	.99683	.99693	.99702	.99711	.9972	.99728	.99736
.99744	.99752	.9976	.99767	.99774	.99781	.99788	.99795	.99801	.99807
.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
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 .70194       .72575     .72907     .73237     .73565       .75804     .76115     .76424     .7673       .81594     .81859     .82121     .82381       .84134     .84375     .84614     .84839       .86433     .8665     .86864     .87076       .88433     .8665     .86864     .87076       .9032     .9049     .90658     .90824       .91924     .92073     .9222     .92364       .93319     .93448     .93574     .93699       .9452     .9463     .94552     .96633       .95431     .95637     .97287     .9732  <tr< td=""><td>0     .01     .02     .03     .04       .5     .50399     .50798     .51197     .51595       .53983     .5438     .54776     .55172     .55567       .57926     .58317     .58706     .59095     .59483       .61791     .62172     .62522     .6293     .63307       .65542     .6591     .66276     .6664     .67003       .61146     .69497     .69847     .70194     .7054       .72575     .72907     .73237     .73565     .73891       .75804     .76115     .76424     .7673     .79955       .81594     .81859     .82121     .82381     .82639       .84134     .84375     .84614     .84849     .85083       .86433     .8665     .86864     .87076     .87286       .88433     .8665     .86864     .87076     .87286       .9032     .9049     .90253     .90824     .92507       .93319     .93448     .93574     .93699     .9322</td><td>0     .01     .02     .03     .04     .05       .53983     .50399     .50798     .51197     .51595     .51994       .53983     .5438     .54776     .55172     .55567     .55962       .57926     .58317     .58706     .59095     .59483     .59871       .61791     .62172     .62552     .6293     .63307     .63683       .55542     .6591     .66644     .67003     .67364       .69146     .69497     .68647     .70194     .7054     .77837       .78814     .79103     .79389     .7673     .77035     .77337       .78814     .79103     .79389     .76673     .77955     .80234       .8159     .82121     .82381     .82639     .82844       .84134     .84375     .84644     .8493     .88636     .87933       .9032     .9049     .9058     .90267     .92647       .91924     .92073     .9222     .92364     .95503       .95543     .95637&lt;</td><td>0     .01     .02     .03     .04     .05     .06       1.5     .50399     .50798     .51197     .51595     .51994     .52392       1.53983     .5438     .54776     .55172     .55567     .55962     .56335       1.57926     .58317     .58706     .59095     .59483     .5438     .64058       .65542     .6591     .66276     .6664     .67003     .67364     .67724       .69146     .69497     .69487     .70194     .7054     .77237     .77637       .75804     .76115     .76424     .7673     .79955     .80234     .80511       .8159     .82121     .82381     .82639     .82944     .8147       .84134     .84375     .84644     .87076     .87286     .87493     .87698       .88433     .8665     .88647     .80824     .9088     .91149     .91309       .9032     .9049     .9058     .9251     .89435     .89617       .9032     .9043     .93679</td><td><math display="block">\begin{array}{cccccccccccccccccccccccccccccccccccc</math></td><td>0     .01     .02     .03     .04     .05     .06     .07     .08       1.5     .50399     .50798     .51197     .51595     .51994     .52392     .5279     .53188       .53983     .5438     .54776     .55172     .55567     .55962     .56356     .56749     .57142       .57926     .58317     .62552     .6293     .63307     .63683     .64058     .64431     .64803       .65542     .6591     .66276     .6664     .6703     .67344     .71266     .71904       .72575     .72907     .73237     .73565     .73891     .74215     .74537     .74857     .75175       .75804     .70113     .79339     .79673     .79955     .80234     .80511     .80785     .81057       .81543     .81859     .82121     .82381     .82639     .82894     .83147     .83988     .83646       .84334     .8465     .8664     .87076     .87286     .87493     .8769     .89973       &lt;</td></tr<></td>	0     .01     .02       5     .50399     .50798       .53983     .5438     .54776       .57926     .58317     .58706       .61791     .62172     .62552       .65542     .65547     .66276       .61146     .69497     .69847       .72575     .72907     .73237       .75804     .76115     .76424       .78144     .79103     .79389       .81594     .81859     .82121       .84134     .84375     .84614       .86433     .8665     .86864       .88493     .88666     .88877       .9032     .9049     .90658       .91924     .92073     .9222       .93319     .93448     .93574       .9452     .9463     .9452       .9543     .95637     .95728       .97128     .97193     .97257       .97128     .97193     .97281       .9861     .98645     .98679       .9821     .98265	0     .01     .02     .03       5     .50399     .50798     .51197       .53983     .5438     .54776     .55172       .57926     .58317     .58706     .59095       .61791     .62172     .62522     .6293       .65542     .6591     .66276     .6664       .6146     .69497     .69847     .70194       .72575     .72907     .73237     .73565       .75804     .76115     .76424     .7673       .81594     .81859     .82121     .82381       .84134     .84375     .84614     .84839       .86433     .8665     .86864     .87076       .88433     .8665     .86864     .87076       .9032     .9049     .90658     .90824       .91924     .92073     .9222     .92364       .93319     .93448     .93574     .93699       .9452     .9463     .94552     .96633       .95431     .95637     .97287     .9732 <tr< td=""><td>0     .01     .02     .03     .04       .5     .50399     .50798     .51197     .51595       .53983     .5438     .54776     .55172     .55567       .57926     .58317     .58706     .59095     .59483       .61791     .62172     .62522     .6293     .63307       .65542     .6591     .66276     .6664     .67003       .61146     .69497     .69847     .70194     .7054       .72575     .72907     .73237     .73565     .73891       .75804     .76115     .76424     .7673     .79955       .81594     .81859     .82121     .82381     .82639       .84134     .84375     .84614     .84849     .85083       .86433     .8665     .86864     .87076     .87286       .88433     .8665     .86864     .87076     .87286       .9032     .9049     .90253     .90824     .92507       .93319     .93448     .93574     .93699     .9322</td><td>0     .01     .02     .03     .04     .05       .53983     .50399     .50798     .51197     .51595     .51994       .53983     .5438     .54776     .55172     .55567     .55962       .57926     .58317     .58706     .59095     .59483     .59871       .61791     .62172     .62552     .6293     .63307     .63683       .55542     .6591     .66644     .67003     .67364       .69146     .69497     .68647     .70194     .7054     .77837       .78814     .79103     .79389     .7673     .77035     .77337       .78814     .79103     .79389     .76673     .77955     .80234       .8159     .82121     .82381     .82639     .82844       .84134     .84375     .84644     .8493     .88636     .87933       .9032     .9049     .9058     .90267     .92647       .91924     .92073     .9222     .92364     .95503       .95543     .95637&lt;</td><td>0     .01     .02     .03     .04     .05     .06       1.5     .50399     .50798     .51197     .51595     .51994     .52392       1.53983     .5438     .54776     .55172     .55567     .55962     .56335       1.57926     .58317     .58706     .59095     .59483     .5438     .64058       .65542     .6591     .66276     .6664     .67003     .67364     .67724       .69146     .69497     .69487     .70194     .7054     .77237     .77637       .75804     .76115     .76424     .7673     .79955     .80234     .80511       .8159     .82121     .82381     .82639     .82944     .8147       .84134     .84375     .84644     .87076     .87286     .87493     .87698       .88433     .8665     .88647     .80824     .9088     .91149     .91309       .9032     .9049     .9058     .9251     .89435     .89617       .9032     .9043     .93679</td><td><math display="block">\begin{array}{cccccccccccccccccccccccccccccccccccc</math></td><td>0     .01     .02     .03     .04     .05     .06     .07     .08       1.5     .50399     .50798     .51197     .51595     .51994     .52392     .5279     .53188       .53983     .5438     .54776     .55172     .55567     .55962     .56356     .56749     .57142       .57926     .58317     .62552     .6293     .63307     .63683     .64058     .64431     .64803       .65542     .6591     .66276     .6664     .6703     .67344     .71266     .71904       .72575     .72907     .73237     .73565     .73891     .74215     .74537     .74857     .75175       .75804     .70113     .79339     .79673     .79955     .80234     .80511     .80785     .81057       .81543     .81859     .82121     .82381     .82639     .82894     .83147     .83988     .83646       .84334     .8465     .8664     .87076     .87286     .87493     .8769     .89973       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  .55962       .57926     .58317     .58706     .59095     .59483     .59871       .61791     .62172     .62552     .6293     .63307     .63683       .55542     .6591     .66644     .67003     .67364       .69146     .69497     .68647     .70194     .7054     .77837       .78814     .79103     .79389     .7673     .77035     .77337       .78814     .79103     .79389     .76673     .77955     .80234       .8159     .82121     .82381     .82639     .82844       .84134     .84375     .84644     .8493     .88636     .87933       .9032     .9049     .9058     .90267     .92647       .91924     .92073     .9222     .92364     .95503       .95543     .95637<	0     .01     .02     .03     .04     .05     .06       1.5     .50399     .50798     .51197     .51595     .51994     .52392       1.53983     .5438     .54776     .55172     .55567     .55962     .56335       1.57926     .58317     .58706     .59095     .59483     .5438     .64058       .65542     .6591     .66276     .6664     .67003     .67364     .67724       .69146     .69497     .69487     .70194     .7054     .77237     .77637       .75804     .76115     .76424     .7673     .79955     .80234     .80511       .8159     .82121     .82381     .82639     .82944     .8147       .84134     .84375     .84644     .87076     .87286     .87493     .87698       .88433     .8665     .88647     .80824     .9088     .91149     .91309       .9032     .9049     .9058     .9251     .89435     .89617       .9032     .9043     .93679	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	0     .01     .02     .03     .04     .05     .06     .07     .08       1.5     .50399     .50798     .51197     .51595     .51994     .52392     .5279     .53188       .53983     .5438     .54776     .55172     .55567     .55962     .56356     .56749     .57142       .57926     .58317     .62552     .6293     .63307     .63683     .64058     .64431     .64803       .65542     .6591     .66276     .6664     .6703     .67344     .71266     .71904       .72575     .72907     .73237     .73565     .73891     .74215     .74537     .74857     .75175       .75804     .70113     .79339     .79673     .79955     .80234     .80511     .80785     .81057       .81543     .81859     .82121     .82381     .82639     .82894     .83147     .83988     .83646       .84334     .8465     .8664     .87076     .87286     .87493     .8769     .89973       <

### Standardizing a Normal Variable

 For a given normal random variable X with mean μ and variance σ<sup>2</sup>, you can standardize it by defining a new random variable Y given by

$$Y = \frac{X - \mu}{\sigma}$$

- Since Y is a form of aX + b, where  $a = \frac{1}{\sigma}$  and  $b = -\frac{\mu}{\sigma}$ , we know that the normality of Y is preserved.
- The mean and variance of Y can be computed as

$$egin{aligned} & E[Y] = rac{E[X]-\mu}{\sigma} = 0 ext{ and} \ & var(Y) = rac{var(X)}{\sigma^2} = 1. \end{aligned}$$

### Example

- Question: The midterm for our CS240 class can be modeled as a normal random variable with a mean of  $\mu = 75\%$  and standard deviation of  $\sigma = 10\%$ . What is the probability that a randomly chosen student has less than or equal to 80% of score?
- Answer: Let X be the score. Then, we first normalize X to have zero mean and unit variance.

$$Y = \frac{X - 75}{10}$$

• Then,

$$P(X < 80) = P\left(Y < \frac{80 - 75}{10}\right) = P\left(Y < \frac{1}{2}\right)$$
$$= \Phi\left(\frac{1}{2}\right)$$
$$= 0.69146.$$

### Example

- Question: What is the probability that a randomly chosen student has greater than or equal to 80% of score?
- Answer:

$$P(X \ge 80) = 1 - P(X < 80) = 1 - \Phi\left(\frac{1}{2}\right) = 0.30854$$